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TREATISE

ON THE

DIFFERENTIAL AND INTEGRAL CALCULUS,

AND THE

CALCULUS OF VARIATIONS.

BY

THOMAS G. HALL, M.A.,

PROFESSOR OF MATHEMATICS IN KING'S COLLEGE, LONDON,

PREBENDARY OF ST PAUL'S;

AND LATE FELLOW AND TUTOR OF MAGDALENE COLLEGE, CAMBRIDGE.

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PREFACE.

A KNOWLEDGE of this branch of the Pure Mathematics is absolutely necessary, before any one can successfully undertake the perusal of works on Natural Philosophy, in which the effects of the observed laws that govern the material world are reduced to calculation.

For Students deficient in this knowledge, yet anxious to obtain as much as may enable them to master the chief analytical difficulties incident to the study of Elementary Treatises on the Mixed Mathematics, this book has been written: and with the hope that by its means, a subject of high interest may be rendered accessible to an increased number of readers.

The Table of Contents which accompanies the work will sufficiently exhibit its plan, as well as the subjects treated of in it.

A few words may be here added, in explanation of the principles adopted in laying down the definitions.

By a method, similar to that of M. Poisson, it is shewn that $u_1 = f(x+h)$ can always be expanded in a series of ascending integral powers of h; which may be written under the convenient form of the equation

$$u_1 = u + Ah + Uh^2.$$

The term Ah, the first term of the difference between $u_1 - u$, is defined to be the differential of u: and A the coefficient of h is called the differential coefficient: and from these definitions, the rules for Differentiation are in general derived.

But since when the general form of $f(x+h)$ has been demonstrated, we see that $\frac{u_1 - u}{h}$ is equal to A, when $h=0$: we may therefore find the differential coefficient, by dividing both sides of the equation $u_1 - u = \phi(x, h)$ (not expanded), by h, and then make $h=0$. This method, which sometimes diminishes the algebraical labour of finding A, is in few instances made use of. It is in fact the method of Limits, often useful in the application of the Calculus, and to which the idea of series is so necessary an auxiliary.

In truth the notion of a series seems inseparably connected with the method of Limits—to which in the Differential Calculus it gives a clearness and precision, of which that method stands much in need. For to say if $u = f(x)$, and u_1 be its value when x becomes x_1, that the differential Coefficient is the value to which $\frac{u_1 - u}{x_1 - x}$ tends while x_1 continually approaches x; without first exhibiting the relation which exists between u_1 and u; is to use a mysterious obscurity which must heavily tax the faith or the credulousness of the reader. But put for u_1 its expansion and the difficulty vanishes.

It has been said that $f(x+h)$ produces diverging series —if by this it is meant, that Ah is improperly determined, the objection lies equally against the method of Limits; and although a preface is not the place to enter into such discussions, it may be remarked that it is one thing to have a diverging series, arising from a known function, and another to determine by a limited number of terms the value of such a series.

If however it be granted that the method of Limits, and that used in our definitions, will when u is a function of one variable only, lead with equal facility to the same re-

sults; this is not the case, with the former method when u is a function of two or a greater number of variables—either some new definition, or some new hypothesis is necessary, in order to arrive at the equation $du = \left(\frac{du}{dx}\right)dx + \left(\frac{du}{dy}\right)dy$, u being a function of x and y; yet how readily is the same result obtained from the expansion of $f(y+k, \ x+h)$. To these remarks I may add, that a long experience has convinced me that the method of series, often combined with and not rejecting that of Limits, is best suited to the instruction of a class.

The symbol $\frac{du}{dx}$ for the differential coefficient of $u = f(x)$, invented by Leibnitz, and used almost without exception by the continental writers, is here retained—I mention the fact, since the notation $d_x u$ for the same term has been revived by some Cambridge Mathematicians.—I do not pretend to decide the question which of the two, $\frac{du}{dx}$ or $d_x u$, estimated by its power of best representing the differential coefficient ought to be preferred, but I see that the latter is, to say the least, an imperfect notation; and is liable to the objection that the suffix x, in the calculus of finite differences has a meaning entirely different from that indicated by the x in d_x. But the most important objection is that already alluded to, namely, that when a student has become acquainted with the proposed notation from his elementary books, his eye must be familiarized with that of Leibnitz, before the works of Lacroix and Laplace can be read with advantage.

Lastly, if it be considered necessary to offer an inducement to any one to enter upon the study of a science—which is the result of one of Newton's most brilliant discoveries, let him know "that it is a high privilege, not a

duty, to study this language of pure unmixed truth. The laws by which God has thought good to govern the universe are surely subjects of lofty contemplation, and the study of that symbolical language by which alone these laws can be fully decyphered, is well deserving of his noblest efforts*."

* Professor Sedgwick on the Studies of the University.

KING'S COLLEGE, LONDON.

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THE DIFFERENTIAL CALCULUS.

CHAPTER I.

1. One quantity u is said to be a function of another x when the value of the magnitude of u depends upon the variation of x. Thus the area of a triangle is a function of the base, when the altitude remains unaltered, since the area will increase or decrease with the increase or decrease of the base.

And if $u = ax^2 \pm bx$, where a and b are constant quantities, and x a variable one, u is said to be a function of x, since if x changes, the value of u will be altered: this relation between u and x is usually expressed by writing $u = f(x)$ or $\phi(x)$, the symbols f and ϕ expressing the word function.

The quantities expressed by the letters a and b are omitted in the equation $u = f(x)$. Since, although they determine the particular kind of function, they remain unchanged, while x passes through every degree of magnitude.

The quantity x is called the independent variable, and u the dependent variable.

2. Functions are called explicit and implicit: u is an explicit function of x, when u is known in terms of x, as in the equation $u = ax^2 + bx$. An implicit function is when u and x are involved together, as in the equation $u^2x - aux + bx^2 = 0$. An implicit function is written $f(u, x)$ or $\phi(u, x) = 0$.

3. Functions are also divided into algebraical and transcendental.

Algebraical functions are those where u may be expressed in terms of x, by means of an equation consisting of a finite number of terms.

Thus $u = ax^m + bx^{m-1} + \text{\&c.} + qx^2 + rx + s$ where (m) is finite, is an *algebraical* function of x.

A *Transcendental* function is one where u is equal to an infinite series, the sum of which cannot be expressed by a limited number of terms.

Thus $u = \log(1+x)$, which

$$= \frac{1}{A}\left\{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \text{\&c. to infinity}\right\},$$

and $u = \sin x = x - \dfrac{x^3}{2.3} + \dfrac{x^5}{2.3.4.5} - \text{\&c. to infinity}.$

are transcendental functions of x*.

4. Functions are also called continuous or discontinuous. A function is continuous, when it undergoes a gradual change; it is discontinuous, when the change is not gradual, or when the function changes suddenly from one value to another very different value. Thus when the difference between $f(x)$ and $f(x+h)$ may, by the continued diminution of h, be made as small as we please, $f(x)$ is a continuous function; but when under the same circumstances $f(x+h)$ differs widely from $f(x)$, the latter is a discontinuous function. We may familiarly liken a continuous quantity, to a stream of water flowing equably and steadily through a tube, and a discontinuous one, to water falling interruptedly, or in drops, from a height.

5. The equation $u = f(x)$ expresses the relation between the function u and the *single* variable x, and the values of u solely depend upon the change that may take place in x: but if we have an equation between three unknown quantities, such as $u = ax^2y - bxy^2$, where x and y are independent of each other, i.e. not connected together by any other equation; then the value of u depends upon the change, both of x and y, and u is said to be a function of two variables; this is expressed by writing $u = f(x, y)$.

As an instance, we may again take the area of a triangle, the magnitude of which depends upon the rectangle of the base and the altitude, which lines are totally independent of each other.

It is obvious that there may be functions of three, four, or of n variables.

6. But to return to functions of one variable; let $u = f(x)$ express the relation between the function and its independent variable x.

Let x increase and become $x+h$, then the value of u

* $u = 1 + x + x^2 + x^3 +$ &c. to infinity is an algebraical function of x, since the sum of the series is expressed by $\dfrac{1}{1-x}$.

will most probably be altered. Let the new value be represented by u_1, then $u_1 = f(x+h)$,

$$\begin{aligned} \text{and} \quad u &= f(x), \text{ by hypothesis;} \\ \hline \therefore\ u_1 - u &= f(x+h) - f(x). \end{aligned}$$

Now $u_1 - u$, or the difference between the functions of $x+h$ and x, must depend upon h, and we shall first shew that it may be expressed by a series of the form

$$Ah + Bh^2 + Ch^3 + \&c.;$$

and $\therefore$ that $u_1 = u + Ah + Bh^2 + Ch^3 + \&c.$,

or that u_1 is equal to u + a series of terms involving positive and integral powers of h, which ascend from the simple power: the primary object of the Differential Calculus is to find the coefficients A, B, C, &c.

7. We will first shew that u_1 may be expressed by a series of the above form by a few particular examples:

(1) Let $u = x^3$;

$$\begin{aligned} \therefore\ u_1 = (x+h)^3 &= x^3 + 3x^2h + 3xh^2 + h^3. \\ &= u + 3x^2h + 3xh^2 + h^3, \end{aligned}$$

which is of the required form.

(2) Next, let $u = x^n$;

$$\therefore\ u_1 = (x+h)^n = x^n + nx^{n-1}h + n\frac{(n-1)}{2}x^{n-2}h^2 + \&c.$$

by the Binomial Theorem,

Or, putting u for x^n,

$$u_1 = u + nx^{n-1}h + n\frac{n-1}{2}x^{n-2}h^2 + \&c.,$$

a series of ascending powers of h.

(3) Let $u = Ax^m + Bx^n + Cx^p + \&c.$;

$$\begin{aligned} \therefore\ u_1 &= A(x+h)^m + B(x+h)^n + C(x+h)^p + \&c. \\ &= A\left\{x^m + mx^{m-1}h + m\frac{(m-1)}{2}x^{m-2}h^2 + \&c.\right\} \\ &\quad + B\left(x^n + nx^{n-1}h + n\frac{(n-1)}{2}x^{n-2}h^2 + \&c.\right) \\ &\quad + C\left(x^p + px^{p-1}h + p\frac{p-1}{2}x^{p-2}h^2 + \&c.\right) \\ &\quad + \&c. \\ &= Ax^m + Bx^n + Cx^p + \&c. \\ &\quad + (mAx^{m-1} + nBx^{n-1} + \&c.)h \end{aligned}$$

$$+\{m\frac{(m-1)}{2}.Ax^{m-2}+n\frac{(n-1)}{2}Bx^{n-2}+\&c.\}h^2$$

$$+\&c. \qquad\qquad +\&c.$$

$$=u+ph+qh^2+\&c.$$

by writing u for its value, $Ax^m+Bx^n+\&c.$, and putting p, q, &c. for the coefficients of h, h^2, &c.

(4) It may also be shewn that a^{x+h}, $\log(x+h)$, $\sin(x+h)$, can be expanded into series of the form

$$u+Ah+Bh^2+Ch^3, \&c.$$

but we proceed to demonstrate the following general Proposition.

8. Prop. If $u=f(x)$, and u_1 be the value of u when x becomes $x+h$, then shall

$$u_1=u+Ah+Uh^2,$$

where u is the original function, and Uh^2, represents all the terms that follow Ah. It is however necessary to prove,

(1) That u_1 or $f(x+h)$ can only contain powers of h with positive indices. For if

$$u_1=M+Ah^\alpha+Bh^{-\beta}+\&c.=M+Ah^\alpha+\frac{B}{h^\beta}+\&c.$$

when $h=0$, u_1 instead of becoming $=u$, would be infinite.

(2) That none of the indices of h can be fractional; for if possible let

$$u_1=M+Ph^{\frac{m}{n}}+R.$$

then $\because h^{\frac{m}{n}}$ or $\sqrt[n]{h^m}$ has n different values, let h_1 and h_2 be two of them;

$$\therefore\ u_1=M+Ph_1+R;\ \text{and}\ u_1=M+Ph_2+R;$$

$\therefore$ subtracting, $P(h_1-h_2)=0$; $\therefore$ $P=0$; or there is no term of the form $Ph^{\frac{m}{n}}$.

(3) That the first term of the expansion $=u$.

For let u_1 or $f(x+h)=M+Ah^\alpha+\&c.$ then let $h=0$;

$\therefore f(x)=u=M$, or $M=u$; and $u_1=u+Ah^\alpha+\&c.$

Let $\therefore$ u_1 or $f(x+h)=u+Ah^\alpha+Bh^\beta+Ch^\gamma+\&c.$,

where α is the least of the indices of h, and β the next in magnitude, and A, B, &c. are functions of x.

Now whether x become $x+h$, or h become $h+h$ or $2h$, u_1 will become $f(x+2h)$, and the expansions on either supposition must be identical.

(1) Let h become $2h$, and let u_2 be the value of u_1;

$$\therefore\ u_2 = f(x+2h)$$
$$= u + A(2h)^\alpha + B(2h)^\beta + \&c.$$
$$= u + 2^\alpha Ah^\alpha + 2^\beta Bh^\beta + \&c. \ldots\ldots (1).$$

(2) Let x become $x+h$, then u_1 is as before $f(x+2h)$ or u_2; also let (u), (A), (B), &c. represent the values of u, A, B, &c., for these undergo change, $\because$ they are functions of x;

$$\therefore\ u_2 = (u) + (A)h^\alpha + (B)h^\beta + \&c.$$

But (u) is the same as u_1, for it is $f(x+h)$,

$$\therefore\ (u) = u + Ah^\alpha + Bh^\beta + \&c.$$

Also (A), (B), &c. being what A, B, &c. become by putting $x+h$ for x, must be of a similar form;

$$\therefore\ (A) = A + A_1h^{\alpha_1} + A_2h^{\beta_1} + \&c.$$
$$(B) = B + B_1h^{\alpha_2} + B_2h^{\beta_2} + \&c.$$

then multiplying (A) by h^α and (B) by h^β, and substituting,

$$\begin{aligned} u_2 = u &+ Ah^\alpha + Bh^\beta + Ch^\gamma + \&c. \\ &+ Ah^\alpha + A_1h^{\alpha+\alpha_1} + A_2h^{\alpha+\beta_1} + \&c. \\ &+ Bh^\beta + B_1h^{\alpha_2+\beta} + \&c. \\ = u &+ 2Ah^\alpha + A_1h^{\alpha+\alpha_1} + 2Bh^\beta + \&c. \ldots (2). \end{aligned}$$

Equating the coefficients of h^α in series (1) and (2),

$$2A = 2^\alpha A;\quad 2 = 2^\alpha;\quad \therefore\ \alpha = 1,$$
$$\text{and } u_1 = f(x+h) = u + Ah + Bh^\beta + \&c.$$

whence it appears that the *second term of the expansion of $f(x+h)$ contains the first power of h only.*

Hence also $\alpha_1 = 1$ for $A_1h^{\alpha_1}$ is the second term of the expression for a when x becomes $x+h$;

$$\therefore\ u_2 = u + 2Ah + A_1h^2 + 2Bh^\beta + \&c. \text{ from } (2)$$
$$= u + 2Ah + 2^\beta . Bh^\beta + \&c. \ldots\ldots\ldots (1).$$

Now, since in series (2) a term is found involving h^2, some corresponding term must be found in series (1); and as β is less than any index that follows it, β must $= 2$;

$$\therefore\ u_1 = f(x+h) = u + Ah + Bh^2 + Ch^\gamma + \&c.$$
$$= u + Ah + (B + Ch^{\gamma-2} + \&c.)h^2$$
$$= u + Ah + Uh^2.$$

Cor. From this it follows that $\beta_1 = 2$; and therefore $\alpha + \beta_1 = 3$; and therefore $A_2h^{\alpha+\beta_1} = A_2h^3$; which being a term in series (2), there must be a corresponding term in series (1); whence $\gamma = 3$, and similarly the next index $= 4$, and so on;

$$\therefore\ u_1 = f(x+h) = u + Ah + Bh^2 + Ch^3 + Dh^4 + \&c.$$

it will afterwards be shewn, by a Theorem, called Taylor's Theorem, that the coefficients, A, B, C, &c. have a dependence on each other.

9. The second term of the expansion, or Ah, is called the differential of u: differential being the diminutive of difference; for Ah is the first term of the difference between u_1 and u, and is consequently a part only of the difference: but the difference and differential differ the less, the less h is, and in cases of approximation, the latter is sometimes taken for the former.

Instead of writing differential at full length, the letter d is used, thus du is put for differential of u, and thus $du = Ah$: but as then h is called the differential of x, therefore for symmetry of notation dx is put for h, and thus $du = Adx$.

A is called the *first differential coefficient*, and is expressed by the symbol $\frac{du}{dx}$, when $u = f(x)$.

Hence we define a *differential* to be *the second term of the* expansion of $f(x+h)$, and the differential coefficient to be the *coefficient of the first power of h.*

The process by which A, or $\frac{du}{dx}$ is found is called *differentiation.*

From these definitions we see that the differential of u, is the product of A into the differential of x; or calling the first quantity δu, and the second δx, we have

$$\delta u = A\delta x\ ;\ \therefore\ \frac{\delta u}{\delta x} = A = \frac{du}{dx},$$

or the ratio of the differentials of u and x is equal to the ratio of the differential coefficient to unity.

The letter δ is here used only to avoid confounding the differentials with the differential coefficient, but in general we make use of the letter d.

10. Again, since $u_1 = u + Ah + Uh^2$,

$$\therefore\ \frac{u_1 - u}{h} = A + Uh\ ;$$

but $u_1 - u$ is the increment of u, and h is the increment of x; therefore the ratio of the increment of the function, to the increment of x, $= A + Uh$; and as h decreases, this ratio tends to A as its limit, and when h vanishes actually $= A$.

That is, A or $\frac{du}{dx}$ is the *limit of the ratio of the increment of the function to that of the variable upon which it depends.*

COR. Hence, the ratio of the differentials of u and x, equals the limit of the ratio of the increments of u and x.

11. Hence we have a method of finding the differential coefficient which is frequently very convenient. Expand $f(x+h)$, subtract $f(x)$, divide both sides by h, make $h=0$; and the term or terms remaining of the expansion will be the coefficient required.

12. We have seen that if u be any function of x, and x become $x+h$,

$$f(x+h) = u + \frac{du}{dx}h + Uh^2.$$

Similarly, if z, v, &c. be functions of x, then they will respectively become when x is made $x+h$,

$$z + \frac{dz}{dx}h + Zh^2, \text{ and } v + \frac{dv}{dx}h + Vh^2,$$

where Zh^2 and Vh^2 represent all the terms after the first two.

13. Thus it appears, that in order to find the differential or differential coefficient, we have merely to put $x+h$ for x, and expand $f(x+h)$ according to the powers of h, and the term corresponding to Ah will give us at once both of the objects of our enquiry. But such a direct process would always be tedious, and often almost impracticable. We therefore proceed to investigate rules which will not only greatly diminish the labour of *differentiation*, but render it a simple algebraical operation; but we will first apply the general process to the function

$$u = \frac{a+x}{b+x};$$

$$\therefore\ u_1 = \frac{a+x+h}{b+x+h} = \frac{\dfrac{a+x}{b+x} + \dfrac{h}{b+x}}{1 + \dfrac{h}{b+x}}$$

$$= \left(\frac{a+x}{b+x} + \frac{h}{b+x}\right)\cdot\frac{1}{1+\dfrac{h}{b+x}}$$

$$= \left(\frac{a+x}{b+x} + \frac{h}{b+x}\right)\cdot\left\{1 - \frac{h}{b+x} + \frac{h^2}{(b+x)^2} - \&c.\right\}$$

$$=\frac{a+x}{b+x}+h\left\{\frac{1}{b+x}-\frac{a+x}{(b+x)^2}\right\}+ph^2+\&c.;$$

$$\therefore\ \frac{du}{dx}=\frac{1}{b+x}-\frac{a+x}{(b+x)^2}=\frac{b-a}{(b+x)^2}.$$

Again, since $u=\frac{a+x}{b+x}$, we shall have by the same process

$$u_1=u+h.\left\{\frac{b-a}{(b+x)^2}\right\}+ph^2+qh^3+\&c.$$

$$\text{and } \frac{u_1-u}{h}=\frac{b-a}{(b+x)^2}+ph+qh^2+\&c.;$$

and by making $h=0$, as in Art. 11,

$$\frac{du}{dx}=\frac{b-a}{(b+x)^2}.$$

Rules for finding the Differential Coefficient.

14. We repeat the definition of Art. 9, that if $u=f(x)$; $\frac{du}{dx}$ is the coefficient of the first power of h in the expansion of u_1, or of $f(x+h)$.

Let $u=ax$, a being a constant quantity;

$$\therefore\ u_1=a(x+h)=ax+ah=u+ah;$$

$$\therefore\ \frac{du}{dx}=a \text{ or } \frac{d(ax)}{dx}=a.$$

Cor. If $u=x$; $\therefore\ \frac{du}{dx}=1$, $\because\ a=1$.

15. Let $u=ax\pm b$, where a and b are constant;

$$\therefore\ u_1=a(x+h)\pm b=ax\pm b+ah=u+ah;$$

$$\therefore\ \frac{du}{dx}=a, \text{ that is, } \frac{d(ax\pm b)}{dx}=a.$$

But by the preceding Article, $\frac{d(ax)}{dx}=a$;

$$\therefore\ \frac{d(ax\pm b)}{dx}=\frac{d(ax)}{dx};$$

that is, constant quantities connected with a variable one by the signs $\pm$ disappear in differentiation.

16. Let $u = ax^m$. Then,

$$u_1 = a(x+h)^m = a.(x^m + mx^{m-1}h + \&c.)$$
$$= ax^m + max^{m-1}.h + \&c.;$$
$$\therefore \frac{du}{dx} = max^{m-1},$$

or to find the differential coefficient of ax^m, *multiply by the index and then diminish the index by unity.*

Ex. $u = 5x^7$; $\therefore \frac{du}{dx} = 35x^6$.

17. Let $u = az$ where z is a function of x;

therefore if x become $x + h$,

$$z \text{ becomes, } z + \frac{dz}{dx}h + Zh^2;$$
$$\therefore u_1 = az + a\frac{dz}{dx}h + aZh^2;$$
$$\therefore \frac{du}{dx} \text{ or } \frac{d(az)}{dx} = a.\frac{dz}{dx}.$$

18. If $u = az + b$, a and b being constant quantities,

$$\text{then } \frac{du}{dx} = a.\frac{dz}{dx},$$
$$\therefore \frac{d(az+b)}{dx} = \frac{adz}{dx} = \frac{d(az)}{dx}.$$

19. Let $u = z + v + w + \&c.$, z, v, w, being functions of x;

$$\therefore u + \frac{du}{dx}h + \&c. = z + \frac{dz}{dx}h + v + \frac{dv}{dx}h + w + \frac{dw}{dx}h + \&c.;$$
$$\therefore \frac{du}{dx} = \frac{dz}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \&c.$$
$$\text{or } \frac{d.(z+v+w+\&c.)}{dx} = \frac{dz}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \&c.$$

Or the differential coefficient of the sum of any functions equals the sum of the differential coefficients of each function.

20. To find the differential coefficient of the product of two functions. Let $u = zv$;

$$\therefore\ u_1 = \left(z + \frac{dz}{dx}h + Zh^2\right)\left(v + \frac{dv}{dx}h + Vh^2\right)$$

$$= zv + \left(z\frac{dv}{dx} + v\,.\,\frac{dz}{dx}\right)h + Bh^2 + \&c.,$$

$$\text{where } B = Zv + Vz + \frac{dz}{dx}\,.\,\frac{dv}{dx};$$

$$\therefore\ \frac{du}{dx} = z\frac{dv}{dx} + v\,.\,\frac{dz}{dx};$$

or the differential coefficient of the product of two functions equals the sum of the products of each function into the differential coefficient of the other.

21. To find the differential coefficient of the quotient of two functions.

Let $u = \frac{z}{v}$; $\therefore\ vu = z$; $v\frac{du}{dx} + u\frac{dv}{dx} = \frac{dz}{dx}$.

$$\therefore\ \frac{du}{dx} = \frac{1}{v}\,.\,\frac{dz}{dx} - \frac{u}{v}\,.\,\frac{dv}{dx}$$

$$= \frac{1}{v}\,.\,\frac{dz}{dx} - \frac{z}{v^2}\,.\,\frac{dv}{dx}$$

$$= \frac{v\,.\,\frac{dz}{dx} - z\,.\,\frac{dv}{dx}}{v^2}.$$

A simple expression, the form of which is more easily remembered than the enunciation.

22. Let $u = zvw$, writing vw for v in Art. 20;

$$\therefore\ \frac{du}{dx} = z\,.\,\frac{d(vw)}{dx} + vw\frac{dz}{dx}.$$

$$\text{But } \frac{d\,.\,(vw)}{dx} = v\,.\,\frac{dw}{dx} + w\,.\,\frac{dv}{dx};$$

$$\therefore\ \frac{du}{dx} = zv\,.\,\frac{dw}{dx} + zw\,.\,\frac{dv}{dx} + vw\,.\,\frac{dz}{dx}.$$

Similarly may the differential coefficient be found for the product of n functions, and it will be equal to the sum of the n products of the differential coefficient of each of the functions multiplied by the remaining $n-1$ functions. Thus,

$$\frac{d\,.\,\{z\,.\,v\,.\,w\,.\,s\ldots(n)\}}{dx} = v\,.\,w\,.\,s\ldots(n-1)\frac{dz}{dx} + zws\ldots(n-1)\,.\,\frac{dv}{dx}$$

$$+ zvs\ldots(n-1)\frac{dw}{dx} + \&c.$$

23. Lemma. If u be a function of z, and z be a function of x, then

$$\frac{du}{dx}=\frac{du}{dz}\frac{dz}{dx}.$$

For if δu, δz, δx be the corresponding differentials of u, z, and x;

$$\text{then } \because \frac{a}{c}=\frac{a}{b}\times\frac{b}{c}; \quad \therefore \frac{\delta u}{\delta x}=\frac{\delta u}{\delta z}.\frac{\delta z}{\delta x}.$$

$$\text{But } \frac{\delta u}{\delta x}=\frac{du}{dx}; \quad \frac{\delta u}{\delta z}=\frac{du}{dz}; \quad \frac{\delta z}{\delta x}=\frac{dz}{dx};$$

$$\therefore \frac{du}{dx}=\frac{du}{dz}\frac{dz}{dx};$$

an important theorem, of which we shall hereafter give another demonstration.

24. Let $u=z^n$, z being $=f(x)$; find $\frac{du}{dx}$.

$$\frac{du}{dz}=nz^{n-1}; \quad \therefore \frac{du}{dx}=\frac{du}{dz}.\frac{dz}{dx}=nz^{n-1}.\frac{dz}{dx};$$

or, to find the differential coefficient of z^n, *multiply by the index, diminish the index by unity, and then multiply by the differential coefficient of* z.

Ex. If $u=(a^2+x^2)^n$ then $z=a^2+x^2$ and $\frac{dz}{dx}=2x$;

$$\therefore \frac{du}{dx}=n(a^2+x^2)^{n-1}.2x.$$

25. The rule for finding the differential coefficient of z^n is perfectly general, but when $n=\frac{1}{2}$ it has a value which it is useful to remember. Thus,

$$\frac{d(\sqrt{z})}{dx}=\frac{1}{2}z^{\frac{1}{2}-1}\frac{dz}{dx}=\frac{\frac{dz}{dx}}{2\sqrt{z}};$$

whence this rule. To find the differential coefficient of the square root of any quantity, divide the differential coefficient of the quantity under the square root, by twice the square root of the quantity itself.

Ex. Let $u=\sqrt{a+bx+cx^2}$;

$$\therefore \frac{du}{dx}=\frac{b+2cx}{2\sqrt{a+bx+cx^2}}.$$

Examples.

(1) $u = 3x^{\frac{7}{3}}$; $\therefore \dfrac{du}{dx} = 3\cdot\frac{7}{3}\cdot x^{\frac{7}{3}-1} = 7x^{\frac{4}{3}}$.

(2) $u = x^3 + x^2 + x + 1$;

$$\frac{du}{dx} = 3x^2 + 2x + 1.$$

(3) $u = (x+a)\cdot(x+b)$;

$$\frac{du}{dx} = (x+a)\frac{d(x+b)}{dx} + (x+b)\cdot\frac{d(x+a)}{dx}. \text{ Art. (20)}$$
$$= x + a + x + b = 2x + (a+b).$$

(4) $u = x(1+x^2)(1+x^3)$;

$$\frac{du}{dx} = (1+x^2)(1+x^3)\frac{dx}{dx} + x(1+x^3)\cdot\frac{d(1+x^2)}{dx} + x(1+x^2)\cdot\frac{d(1+x^3)}{dx}$$
$$= (1+x^2)(1+x^3) + x(1+x^3)\cdot 2x + x(1+x^2)3x^2$$
$$= 1 + 3x^2 + 4x^3 + 6x^5.$$

(5) $u = \dfrac{a^n}{x^n} = a^n x^{-n}$; $\dfrac{du}{dx} = -na^n x^{-n-1} = -\dfrac{na^n}{x^{n+1}}$;

$$\therefore \text{ if } u = \frac{a}{x},\ \frac{du}{dx} = -\frac{a}{x^2}.$$

(6) $u = \dfrac{x+a}{x+b}$; See (Art. 13).

$$\therefore \frac{du}{dx} = \frac{(x+b)\cdot\dfrac{d(x+a)}{dx} - (x+a)\cdot\dfrac{d(x+b)}{dx}}{(x+b)^2}$$
$$= \frac{x+b-(x+a)}{(x+b)^2} = \frac{b-a}{(x+b)^2}.$$

(7) $u = \dfrac{x^m}{(x+1)^m}$;

$$\therefore \frac{du}{dx} = \frac{(x+1)^m\cdot mx^{m-1} - x^m\cdot m\cdot \overline{x+1}|^{m-1}}{(x+1)^{2m}}$$
$$= \frac{(x+1)\cdot mx^{m-1} - mx^m}{\overline{x+1}|^{m+1}} = \frac{mx^{m-1}}{\overline{x+1}|^{m+1}}.$$

(8) $u = \sqrt{1+x^2}$;

$$\frac{du}{dx} = \frac{2x}{2\sqrt{1+x^2}} = \frac{x}{\sqrt{1+x^2}}. \text{ Art. (25).}$$

(9) $$u=\sqrt{x+\sqrt{1+x^2}};\quad \therefore\ u^2=x+\sqrt{1+x^2};$$

$$\therefore\ 2u\frac{du}{dx}=1+\frac{x}{\sqrt{1+x^2}}=\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}=\frac{u^2}{\sqrt{1+x^2}};$$

$$\therefore\ \frac{du}{dx}=\tfrac{1}{2}\cdot\frac{\sqrt{x+\sqrt{1+x^2}}}{\sqrt{1+x^2}}.$$

(10) $$u=\frac{\sqrt{a^2+x^2}}{\sqrt{a^2-x^2}};\quad \therefore\ u^2=\frac{a^2+x^2}{a^2-x^2};$$

$$\therefore\ 2u\frac{du}{dx}=\frac{2x(a^2-x^2)+2x(a^2+x^2)}{(a^2-x^2)^2}=\frac{4a^2x}{(a^2-x^2)^2};$$

$$\therefore\ \frac{du}{dx}=\frac{2a^2x}{(a^2-x^2)^2}\,\frac{\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}}=\frac{2a^2x}{(a^2-x^2)^{\frac{3}{2}}\sqrt{a^2+x^2}}.$$

(11) $$u=\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}=\frac{(\sqrt{1+x}+\sqrt{1-x})^2}{2x}$$

$$=\frac{1+\sqrt{1-x^2}}{x};$$

$$\therefore\ \frac{du}{dx}=\frac{\dfrac{-x^2}{\sqrt{1-x^2}}-(1+\sqrt{1-x^2})}{x^2}=-\frac{x^2+\sqrt{1-x^2}+1-x^2}{x^2\sqrt{1-x^2}}$$

$$=-\frac{1+\sqrt{1-x^2}}{x^2\sqrt{1-x^2}}.$$

(12) $$u=x(1+x^2)\cdot\sqrt{1-x^2}=(x+x^3)\sqrt{1-x^2}$$

$$\frac{du}{dx}=(1+3x^2)\sqrt{1-x^2}-\frac{x^2+x^4}{\sqrt{1-x^2}}$$

$$=\frac{1+3x^2-x^2-3x^4-x^2-x^4}{\sqrt{1-x^2}}$$

$$=\frac{1+x^2-4x^4}{\sqrt{1-x^2}}.$$

(13) $$u=(2x+3)(5x+7);\quad \frac{du}{dx}=20x+29.$$

(14) $$u=(1-x^2)(1-x^3);\quad \frac{du}{dx}=-2x-3x^2+5x^4.$$

(15) $$u=(x^4-x^2+1)^{12};\quad \frac{du}{dx}=24x(x^4-x^2+1)^{11}(2x^2-1).$$

(16) $u=(2ax+x^2)^m$; $\dfrac{du}{dx}=2m(a+x)(2ax+x^2)^{m-1}$.

(17) $u=x^2(1+x^4)$; $\dfrac{du}{dx}=2x(1+3x^4)$.

(18) $u=(1+2x^2)(1+4x^3)$; $\dfrac{du}{dx}=4x(1+3x+10x^3)$.

(19) $u=2x(1+x)^{\frac{5}{3}}$; $\dfrac{du}{dx}=(2+7x)(1+x)^{\frac{2}{3}}$.

(20) $u=(1+x)^4(1+x^2)^2$;

$$\frac{du}{dx}=4(1+x)^3(1+x^2)\{1+x+2x^2\}.$$

(21) $u=(a+x)(b+x)(c+x)$;

$$\frac{du}{dx}=3x^2+2(a+b+c)x+ab+ac+bc.$$

(22) $u=(1-2x)(1-3x)(1-4x)$;

$$\therefore \frac{du}{dx}=-(9-52x+72x^2).$$

(23) $u=(1+x^2)^3(1+x^3)^2$

$$\frac{du}{dx}=6x(1+x^2)^2(1+x^3)(1+x+2x^3).$$

(24) $u=(1+x^m)^n(1+x^n)^m$.

$$\therefore \frac{du}{dx}=mn(1+x^m)^{n-1}(1+x^n)^{m-1}\{x^{m-1}+x^{n-1}+2x^{m+n-1}\}.$$

(25) $u=(1+x)\sqrt{1-x^2}$;

$$\frac{du}{dx}=\frac{1-x-2x^2}{\sqrt{1-x^2}};$$

(26) $u=\dfrac{x^4-1}{x^4+1}$; $\dfrac{du}{dx}=\dfrac{8x^3}{(x^4+1)^2}$.

(27) $u=\dfrac{x}{\sqrt{1+x^2}}$; $\dfrac{du}{dx}=\dfrac{1}{(1+x^2)^{\frac{3}{2}}}$.

(28) $u=\dfrac{x^2-2}{3}\sqrt{1+x^2}$; $\dfrac{du}{dx}=\dfrac{x^3}{\sqrt{1+x^2}}$.

(29) $u=x^{\frac{1}{3}}\sqrt{x^{\frac{1}{2}}+1}$; $\dfrac{du}{dx}=\dfrac{7x^{\frac{1}{2}}+4}{12\sqrt[3]{x^2}\sqrt{x^{\frac{1}{2}}+1}}$.

(30) $u=\dfrac{x^2}{\sqrt{1+x^4}}$; $\dfrac{du}{dx}=\dfrac{2x}{(1+x^4)^{\frac{3}{2}}}$.

(31) $u=\dfrac{x^4-x^2+1}{x^4+x^2+1}$; $\dfrac{du}{dx}=\dfrac{4x(x^4-1)}{(x^4+x^2+1)^2}$.

(32) $u = \dfrac{x^3}{\sqrt{1+x^6}}$; $\dfrac{du}{dx} = \dfrac{3x^2}{(1+x^6)^{\frac{3}{2}}}$.

(33) $u = \dfrac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}$; $\dfrac{du}{dx} = \dfrac{2(\sqrt{1+x^2}-1)^2}{x^3\sqrt{1+x^2}}$.

(34) $u = \dfrac{(x+1)^{\frac{3}{2}}}{\sqrt{x-1}}$; $\dfrac{du}{dx} = \dfrac{(x-2)\sqrt{x+1}}{(x-1)^{\frac{3}{2}}}$.

(35) $u = \sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}$; $\dfrac{du}{dx} = -\dfrac{1}{2(1+\sqrt{x})\sqrt{x-x^2}}$.

(36) $u = \dfrac{1+2x}{\sqrt{1+x+x^2}}$; $\dfrac{du}{dx} = \dfrac{3}{2(1+x+x^2)^{\frac{3}{2}}}$.

(37) $u = \dfrac{\sqrt{1-x+x^2}}{\sqrt{1+x+x^2}}$; $\dfrac{du}{dx} = \dfrac{x^2-1}{(1-x+x^2)^{\frac{1}{2}}(1+x+x^2)^{\frac{3}{2}}}$.

(38) Let $u^2x - u + x^2 - a^2 = 0$; find $\dfrac{du}{dx}$, this is an implicit function; put $v = u^2x$, and differentiate;

$$\therefore \frac{dv}{dx} - \frac{du}{dx} + 2x = 0: \text{ but } \frac{dv}{dx} = 2ux\frac{du}{dx} + u^2 \quad (20);$$

$$\therefore 2ux\frac{du}{dx} + u^2 - \frac{du}{dx} + 2x = 0;$$

$$\therefore \frac{du}{dx}(2ux-1) = -(u^2+2x) = -\frac{u+x^2+a^2}{x}.$$

$$\therefore \frac{du}{dx} = -\frac{u+x^2+a^2}{2ux^2-x}.$$

(39) $u^3 - 3ux^2 + x^3 = 0$; find $\dfrac{du}{dx}$;

$$\therefore 3u^2\frac{du}{dx} - 3x^2\frac{du}{dx} - 6ux + 3x^2 = 0;$$

$$\therefore \frac{du}{dx} = \frac{2ux-x^2}{u^2-x^2} = \frac{u}{x}.$$

(40) $2ux + au^2 - bx^2 = 0$; $\dfrac{du}{dx} = \dfrac{u}{x}$.

(41) $au^4 + c^3ux - bx^4 = 0$; $\dfrac{du}{dx} = \dfrac{u}{x}\left(\dfrac{4au^3+3c^3x}{4au^3+c^3x}\right)$.

(42) $ux = (a+u)\sqrt{b^2-u^2}$; $\dfrac{du}{dx} = -\dfrac{u^3x}{(a+u)(ab^2+u^3)}$.

(43) $u = \sqrt{a+x+\sqrt{a+x+\sqrt{a+x}}}$ &c. *in infin.*;

$$\therefore \frac{du}{dx} = \frac{1}{\sqrt{1+4a+4x}}.$$

26. In the preceding examples, differential coefficients have only been obtained; but by the definitions, the differentials may be found by multiplying the differential coefficients by the increment of the independent variable.

Thus to find the differential of the product of two functions z and v, if $u = zv$, then

$$\frac{du}{dx} = z\frac{dv}{dx} + v\frac{dz}{dx}; \text{ let } \delta x = \text{increment of } x;$$

$$\therefore d(zv) = z\frac{dv}{dx}\delta x + v\frac{dz}{dx}\delta x.$$

$$\text{But } \frac{dv}{dx}\delta x = \text{differential of } v = dv,$$

$$\text{and } \frac{dz}{dx}\delta x = \text{differential of } z = dz;$$

$$\therefore d(zv) = zdv + vdz;$$

and in the same manner,

$$d.\left(\frac{z}{v}\right) = \frac{vdz - zdv}{v^2},$$

$$\text{and } d.(z^n) = nz^{n-1}dz.$$

27. It will now be natural for the student to enquire what is the object to be attained, by finding the differential coefficient, but it will be difficult at present to give a completely satisfactory answer to the enquiry, without introducing subjects with which he can have no acquaintance. Lacroix says: "Il serait fort difficile d'expliquer clairement la nature du Calcul différential à ceux qui n'en ont pas les premières notions." Yet perhaps he may be told, that if y be the ordinate and x the abscissa of a curve, $\frac{dy}{dx}$ is the trigonometrical tangent of the angle, which the tangent makes with the axis of x; that if u be the area of the same curve, $\frac{du}{dx} = y$; an equation by which hereafter the area of the curve may be found. Again, if s represent the space described by a point in a time t, that $\frac{ds}{dt}$ represents the velocity (v) with which at the end of that time the point is moving, and $\frac{dv}{dt}$, the force which either accelerates or retards the

point's motion. And again, if $u=f(x)$ be an equation admitting of maximum or minimum values, that $\frac{du}{dx}$ will furnish an equation, by which the values of x that make u a maximum or minimum may be found. And lastly, if $u=f(x)=0$ be an algebraic equation of which the roots are a, b, c, &c., then $\frac{du}{dx}=0$ will give the limiting equation, the knowledge of the roots of which is so useful in determining the roots of the original equation.

28. We shall conclude this Chapter by a few simple applications.

(1) The radius of a circular plate of metal is 12 inches; find the increase of area when the radius is increased .001 inch.

If $u=$ area of a circle, radius $=x$;

$$\therefore\ u=\pi x^2;\ \text{and } du=2\pi x dx.$$

Make $x=12$; $dx=.001$; then $du=$ increase of area;

$\therefore\ du=3.1416\times 24\times .001=.0753984$ of a square inch.

(2) A cube of metal of the same thickness is similarly increased; find the cubical expansion.

$u=x^3$; $\therefore\ du=3x^2dx=3\times 144\times .001=.432$ cubic inch.

COR. Divide by u; $\therefore\ \frac{du}{u}=\frac{3dx}{x}$. Now $\frac{du}{u}$ is in chemistry called the cubical, and $\frac{dx}{x}$ the linear expansion; hence the cubical is three times the linear expansion.

(3) As an instance of finding the ratio of infinitely small quantities. Upon AB describe a semicircle, draw a chord AP; draw PN perpendicular to AB; then prove that $AP=PN$ ultimately; i.e. at the moment when the arc AP vanishes.

Make $AN=x$, $AB=2a$;

$$\therefore\ AP=\sqrt{2ax},\ PN=\sqrt{2ax-x^2};$$

$$\therefore\ \frac{AP}{PN}=\frac{\sqrt{2ax}}{\sqrt{2ax-x^2}}=\frac{\sqrt{2a}}{\sqrt{2a-x}};\ \text{and if } x=0,$$

$$\frac{AP}{PN}=\frac{\sqrt{2a}}{\sqrt{2a}}=1;\ \text{or } AP=PN \text{ ultimately.}$$

(4) The limit of the ratio of $\sin x : \sin\frac{x}{2}$ is $2:1$.

CHAPTER II.

Differentiation of Angular, Exponential, and Logarithmic Functions.

29. To find the differential coefficient of u, when

$$u = \sin x, \quad \cos x, \quad \tan x, \quad \sec x, \quad \&c.$$

The following Proposition must first be proved.

If h be an angle, $\frac{\sin h}{h}$, or $\frac{\tan h}{h}$ = unity, when $h = 0$.

It is known, that $h > \sin h$, $< \tan h$ (*Trig.* Art. 57)

or h lies between $\sin h$ and $\tan h$,

or $\sin h$, h, and $\tan h$ are in order of magnitude;

$\therefore \tan h - \sin h$, is $> h - \sin h$, or $> \tan h - h$.

If therefore $\tan h - \sin h$ ever $= 0$, or $\frac{\tan h}{\sin h} = 1$; *a fortiori* will $h - \sin h = 0$, and $\tan h - h = 0$; or $\frac{\sin h}{h} = \frac{\tan h}{h} = 1$.

Now $\frac{\sin h}{\tan h} = \frac{\cos h}{1} = \frac{1}{1} = 1$, when $h = 0$;

$\therefore \frac{\sin h}{h}$ and $\frac{\tan h}{h}$ also respectively $= 1$, if $h = 0$.

30. Let $u = \sin x$; find $\frac{du}{dx}$.

For x, put $x + h$, $\therefore u$ becomes $u + \frac{du}{dx}h + Uh^2$,

$$\text{and} \quad u + \frac{du}{dx}h + Uh^2 = \sin(x + h),$$

$$\text{and} \quad u = \sin x;$$

$$\therefore \frac{du}{dx}h + Uh^2 = \sin(x + h) - \sin x$$

$$= 2\cos(x + \tfrac{1}{2}h) \cdot \sin \tfrac{1}{2}h\text{*}.$$

$$\therefore \frac{du}{dx} + Uh = \cos(x + \tfrac{1}{2}h) \cdot \frac{\sin \frac{1}{2}h}{\frac{1}{2}h};$$

* Since $\sin A - \sin B = 2\cos . \frac{A+B}{2} . \sin \frac{A-B}{2}$.

and making $h = 0$; $\therefore \dfrac{\sin \frac{1}{2}h}{\frac{1}{2}h} = 1$;

$$\therefore \frac{du}{dx} = \cos x, \text{ or } \frac{d \, . \sin x}{dx} = \cos x.$$

31. $u = \cos x$; find $\dfrac{du}{dx}$;

$$\therefore u + \frac{du}{dx} h + Uh^2 = \cos (x + h);$$

$$\therefore \frac{du}{dx} h + Uh^2 = \cos (x + h) - \cos x$$

$$= -2 \sin (x + \tfrac{1}{2}h) \sin \tfrac{1}{2}h;$$

$$\therefore \frac{du}{dx} + Uh = -\sin (x + \tfrac{1}{2}h) \frac{\sin \frac{1}{2}h}{\frac{1}{2}h};$$

$$\therefore \text{making } h = 0, \; \frac{du}{dx} = \frac{d \, . \cos x}{dx} = -\sin x.$$

32. $u = \tan x$; find $\dfrac{du}{dx}$;

$$\therefore u + \frac{du}{dx} h + Uh^2 = \tan (x + h);$$

$$\therefore \frac{du}{dx} h + Uh^2 = \tan (x + h) - \tan x$$

$$= \frac{\tan h (1 + \tan^2 x)}{1 - \tan x \, . \tan h};$$

$$\therefore \frac{du}{dx} + Uh = \frac{\tan h}{h} \cdot \frac{(1 + \tan^2 x)}{1 - \tan x \, . \tan h};$$

make $h = 0$; $\therefore \tan h = 0$, and $\dfrac{\tan h}{h} = 1$;

$$\therefore \frac{du}{dx} = \frac{d \, . \tan x}{dx} = 1 + \tan^2 x = \sec^2 x = \frac{1}{\cos^2 x}.$$

33. $u \sec x = \dfrac{1}{\cos x}$; find $\dfrac{du}{dx}$;

$$\therefore \frac{du}{dx} = \frac{\frac{-d \, . \cos x}{dx}}{(\cos x)^2} = \frac{\sin x}{(\cos x)^2} = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x},$$

$$\text{or } \frac{d \, . \sec x}{dx} = \tan x \, . \sec x.$$

34. $u = V.\sin x = 1 - \cos x$;

$$\therefore \frac{du}{dx} = -\frac{d.\cos x}{dx} = \sin x.$$

35. $u = \text{cotan}\, x = \frac{\cos x}{\sin x}$.

$$\frac{du}{dx} = -\frac{(\sin x)^2 + (\cos x)^2}{(\sin x)^2}$$

$$= -\frac{1}{(\sin x)^2} = -(\text{cosec}\, x)^2 = -(1 + \cot^2 x).$$

36. $u = \text{cosec}\, x = \frac{1}{\sin x}$;

$$\therefore \frac{du}{dx} = \frac{-\frac{d\sin x}{dx}}{(\sin x)^2} = \frac{-\cos x}{(\sin x)^2} = -\cot x.\text{cosec}\, x.$$

37. Hence collecting the results,

If $u = \sin x$, $\qquad \frac{du}{dx} = \cos x$,

$u = \cos x$, $\qquad \frac{du}{dx} = -\sin x$,

$u = \tan x$, $\qquad \frac{du}{dx} = 1 + \tan^2 x = \frac{1}{\cos^2 x}$,

$u = \sec x$, $\qquad \frac{du}{dx} = \sec x.\tan x$,

$u = v.\sin x$; $\therefore \frac{du}{dx} = \sin x$,

$u = \cot x$, $\qquad \frac{du}{dx} = -(1 + \cot^2 x) = -\frac{1}{\sin^2 x}$,

$u = \text{cosec}\, x$, $\qquad \frac{du}{dx} = -\text{cosec}\, x.\cot x$.

38. Next let $u = \sin z$, where $z = f(x)$.

$$\text{Then } \frac{du}{dx} = \frac{du}{dz}.\frac{dz}{dx}.$$

$$\text{But } \frac{du}{dz} = \cos z; \quad \therefore \frac{du}{dx} = \cos z \frac{dz}{dx}.$$

39. Let $u = \cos z$; find $\frac{du}{dx}$.

$$\frac{du}{dz} = -\sin z; \quad \therefore \frac{du}{dx} = -\sin z.\frac{dz}{dx}.$$

Ex. (1) Let $u = \sin 3x$; $\therefore \dfrac{du}{dx} = 3\cos 3x$.

(2) Let $u = \cos(ax + b)$; $\therefore \dfrac{du}{dx} = -a \cdot \sin(ax + b)$.

40. Let $u = \tan z$; $\therefore \dfrac{du}{dz} = 1 + \tan^2 z$;

$$\therefore \frac{du}{dx} = (1 + \tan^2 z)\frac{dz}{dx}.$$

41. And if $u = \sec z$; $\quad \dfrac{du}{dx} = \sec z \cdot \tan z \cdot \dfrac{dz}{dx}$,

$u = v \sin z$; $\quad \dfrac{du}{dx} = \sin z \cdot \dfrac{dz}{dx}$,

$u = \cot z$; $\quad \dfrac{du}{dx} = -(1 + \cot^2 z)\dfrac{dz}{dx}$,

$u = \operatorname{cosec} z$; $\quad \dfrac{du}{dx} = -\operatorname{cosec} z \cdot \cot z \cdot \dfrac{dz}{dx}$.

42. To find the differential coefficients of the angle in terms of the sine, cosine, tangent, &c.

Before we do this it will be necessary to shew that if u be a function of x, or if $u = f(x)$, and consequently x a function of u (since it is a matter of convention which of the two is the independent variable), or as it is written $f^{-1}(u)$, where f^{-1} is called the inverse function,

$$\frac{du}{dx} = \frac{1}{\frac{dx}{du}}.$$

Let δu, δx, be the differentials of u and x.

Then since $\dfrac{a}{b} = \dfrac{1}{\frac{b}{a}}$; $\quad \therefore \dfrac{\delta u}{\delta x} = \dfrac{1}{\frac{\delta x}{\delta u}}$.

But since the ratio of the differentials is equal to the ratio of the differential coefficient to unity;

$$\therefore \frac{\delta u}{\delta x} = \frac{du}{dx}; \quad \frac{\delta x}{\delta u} = \frac{dx}{du}. \quad \text{And } \therefore \frac{du}{dx} = \frac{1}{\frac{dx}{du}}.$$

43. Hence*, if $u = \sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, &c. find $\dfrac{du}{dx}$.

* By $u = \sin^{-1} x$ is meant, u is an angle whose sine is x. Similarly, $u = \tan^{-1} x$ is an angle u of which the tangent is x; these are called in-

(1) $u = \sin^{-1} x$; $\therefore\ x = \sin u$;

$$\therefore\ \frac{dx}{du} = \cos u = \sqrt{1 - \sin^2 u} = \sqrt{1 - x^2};$$

$$\therefore\ \frac{du}{dx} = \frac{1}{\frac{dx}{du}} = \frac{1}{\sqrt{1 - x^2}}.$$

(2) $u = \cos^{-1} x$, or $x = \cos u$; $\therefore\ \frac{dx}{du} = -\sin u$;

$$\therefore\ \frac{du}{dx} = -\frac{1}{\sin u} = -\frac{1}{\sqrt{1 - x^2}}.$$

(3) $u = \tan^{-1} x$; $\therefore\ x = \tan u$; $\therefore\ \frac{dx}{du} = (1 + \tan^2 u)$;

$$\therefore\ \frac{du}{dx} = \frac{1}{1 + \tan^2 u} = \frac{1}{1 + x^2}.$$

(4) $u = \sec^{-1} x$; $\therefore\ x = \sec u$; $\therefore\ \frac{dx}{du} = \sec u \tan u$;

$$\therefore\ \frac{du}{dx} = \frac{1}{\sec u \tan u} = \frac{1}{x\sqrt{x^2 - 1}}.$$

(5) $u = \cot^{-1} x$; $\therefore\ x = \cot u$; $\therefore\ \frac{dx}{du} = -(1 + \cot^2 u)$;

$$\therefore\ \frac{du}{dx} = -\frac{1}{1 + \cot^2 u} = -\frac{1}{1 + x^2}.$$

(6) $u = \operatorname{cosec}^{-1} x$; $\therefore\ x = \operatorname{cosec} u$; $\therefore\ \frac{dx}{du} = -\operatorname{cosec} u \,.\, \cot u$;

$$\therefore\ \frac{du}{dx} = \frac{-1}{\operatorname{cosec} u \cot u} = -\frac{1}{x\sqrt{x^2 - 1}}.$$

(7) $u = v \,.\, \sin^{-1} x$; $\therefore\ x = v \sin u$;

$$\therefore\ \frac{dx}{du} = \sin u = \sqrt{1 - \cos^2 u} = \sqrt{(1 - \cos u)(1 + \cos u)}.$$

But $1 - \cos u = x$; $1 + \cos u = 2 - x$,

$$\therefore\ \frac{dx}{du} = \sqrt{2x - x^2} \text{ and } \frac{du}{dx} = \frac{1}{\sqrt{2x - x^2}}.$$

verse functions. Thus, if $u = \log x$, then $u = \log^{-1} x$ expresses that u is a number of which the logarithm is x.

Hence
$$\frac{d\,.\sin^{-1}x}{dx}=\frac{1}{\sqrt{1-x^2}},$$
$$\frac{d\,.\cos^{-1}x}{dx}=\frac{-1}{\sqrt{1-x^2}},$$
$$\frac{d\,.\tan^{-1}x}{dx}=\frac{1}{1+x^2},$$
$$\frac{d\,.\sec^{-1}x}{dx}=\frac{1}{x\sqrt{x^2-1}},$$
$$\frac{d\,.\cot^{-1}x}{dx}=\frac{-1}{1+x^2},$$
$$\frac{d\,.\operatorname{cosec}^{-1}x}{dx}=\frac{-1}{x\sqrt{x^2-1}},$$
$$\frac{d\,.v\sin^{-1}x}{dx}=\frac{1}{\sqrt{2x-x^2}}.$$

44. Again, if $u=\sin^{-1}\frac{x}{a}$; $\therefore\ \frac{x}{a}=\sin u$;

$$\therefore\ \frac{dx}{du}=a\cos u=a\sqrt{1-\frac{x^2}{a^2}}=\sqrt{a^2-x^2};$$

(1) $$\therefore\ \frac{du}{dx}=\frac{1}{\frac{dx}{du}}=\frac{1}{\sqrt{a^2-x^2}},$$

(2) if $u=\cos^{-1}\frac{x}{a}$, $\frac{du}{dx}=-\frac{1}{\sqrt{a^2-x^2}}$.

(3) If $u=\tan^{-1}\frac{x}{a}$; $\therefore\ \frac{x}{a}=\tan u$,

$$\frac{dx}{du}=a(1+\tan^2 u)=a\left(1+\frac{x^2}{a^2}\right)=\frac{a^2+x^2}{a};$$

$$\therefore\ \frac{d\left(\tan^{-1}.\frac{x}{a}\right)}{dx}=\frac{a}{a^2+x^2}.$$

(4) Similarly, $$\frac{d.\left(\sec^{-1}\frac{x}{a}\right)}{dx}=\frac{a}{x\sqrt{x^2-a^2}}.$$

45. Also, if $u=\sin^{-1}z$, where $z=f(x)$, to find $\frac{du}{dx}$.

$$\frac{du}{dz} = \frac{1}{\sqrt{1-z^2}}; \quad \therefore \frac{du}{dx} = \frac{du}{dz} \cdot \frac{dz}{dx} = \frac{\frac{dz}{dx}}{\sqrt{1-z^2}}.$$

Let $u = \cos^{-1} z$; $\therefore \dfrac{du}{dz} = \dfrac{-1}{\sqrt{1-z^2}}$; $\therefore \dfrac{du}{dx} = \dfrac{-\frac{dz}{dx}}{\sqrt{1-z^2}}$.

Let $u = \tan^{-1} z$; $\therefore \dfrac{du}{dz} = \dfrac{1}{1+z^2}$; $\therefore \dfrac{du}{dx} = \dfrac{\frac{dz}{dx}}{1+z^2}$,

and in the same manner for the other circular functions.

46. To find the differentials from the differential coefficients:

$$d\,.\,(\sin x) = \cos x\,.\,dx,$$
$$d\,.\,(\cos x) = -\sin x\,.\,dx,$$
$$d\,.\,(\tan x) = (1 + \tan^2 x)\,dx,$$
$$\text{and } d\,.\,(\sin^{-1} x) = \frac{dx}{\sqrt{1-x^2}},$$
$$d\,.\,(\tan^{-1} x) = \frac{dx}{1+x^2},$$
$$d\,.\,(\sec^{-1} x) = \frac{dx}{x\sqrt{x^2-1}}.$$

Ex. Find that angle (x) which increases twice as fast as its sine.

Let $u = \sin x$; $du = \cos x\,.\,dx$.

But $du = \frac{1}{2}dx$; $\therefore \cos x = \frac{1}{2}$; $\therefore x = 60^\circ$.

Exponential and Logarithmic Functions.

47. Let $u = a^x$, which in general expresses the relation between a number u and its logarithm x, find $\dfrac{du}{dx}$.

Since $u = a^x$; $\therefore u_1 = a^{x+h} = a^x\,.\,a^h$.

But $a^h = 1 + Ah + \frac{1}{2}A^2h^2 + \&c.$ where $A = \log_e a$: (Alg. 269);

$$\therefore u_1 = a^x(1 + Ah + \tfrac{1}{2}A^2h^2 + \&c.);$$

$$\therefore \frac{du}{dx} = Aa^x \text{ and } du = Aa^x\,.\,dx.$$

COR. If $a=e=2.71828$, $A=\log_e e=1$;

$$\therefore \frac{d.e^x}{dx}=e^x; \text{ and } d.e^x=e^x dx.$$

48. Next let $u=\log x$; $\therefore x=a^u$; $\therefore \frac{dx}{du}=Aa^u=A.x$;

$$\therefore \frac{du}{dx}=\frac{1}{\frac{dx}{du}}=\frac{1}{A}.\frac{1}{x}.$$

If the base be (e), $A=1$ and $\frac{du}{dx}=\frac{1}{x}$, or $d.\log x=\frac{dx}{x}$.

49. Again, if $u=a^z$, find $\frac{du}{dx}$.

$$\frac{du}{dz}=Aa^z, \therefore \frac{du}{dx}=\frac{du}{dz}.\frac{dz}{dx}=A.a^z.\frac{dz}{dx}.$$

COR. If $a=e$, or $A=1$, $\therefore \frac{d.e^z}{dx}=e^z.\frac{dz}{dx}$.

50. If $u=\log(z)$, find $\frac{du}{dx}$.

$$\frac{du}{dz}=\frac{1}{A}.\frac{1}{z}; \quad \therefore \frac{du}{dx}=\frac{du}{dz}.\frac{dz}{dx}=\frac{1}{A}.\frac{1}{z}.\frac{dz}{dx}.$$

If $A=1$, $\frac{d.(\log z)}{dx}=\frac{\frac{dz}{dx}}{z}$; and $d.(\log z)=\frac{dz}{z}$.

From the former of which equations, we obtain this rule:

The differential coefficient of the logarithm of a function equals the differential coefficient of the function, divided by the function itself.

Ex. $u=\log\sqrt{x^4+x^2+1}$

$$=\tfrac{1}{2}\log(x^4+x^2+1): \frac{du}{dx}=\frac{2x^3+x}{x^4+x^2+1}.$$

Observe in future, whenever *log* is used, the Napierian logarithm is meant.

Examples.

(1) $u=(\sin x)^n$; $\frac{du}{dx}=n(\sin x)^{n-1}\cos x$.

(2) $u=\sin nx$; $\frac{du}{dx}=n\cos nx$.

(3) $u = (\tan x)^3$; $\dfrac{du}{dx} = 3\tan^2 x \sec^2 x$.

(4) $u = \sin 3x \cdot \cos 2x$,

$$\frac{du}{dx} = 3\cos 3x \cos 2x - 2\sin 3x \cdot \sin 2x$$
$$= \cos 3x \cos 2x + 2(\cos 3x \cos 2x - \sin 3x \sin 2x)$$
$$= \cos 3x \cos 2x + 2\cos 5x.$$

(5) $u = \sin(\cos x) = \sin z$, if $z = \cos x$;

$$\therefore \frac{du}{dx} = \frac{du}{dz} \cdot \frac{dz}{dx} = \cos z \cdot (-\sin x) = -\sin x \cos(\cos x).$$

(6) $u = \sin^{-1} \dfrac{x}{\sqrt{1+x^2}} = \sin^{-1} z$, if $z = \dfrac{x}{\sqrt{1+x^2}}$,

$$\frac{du}{dx} = \frac{\frac{dz}{dx}}{\sqrt{1-z^2}},$$

$$\frac{dz}{dx} = \frac{\sqrt{1+x^2} - \frac{x^2}{\sqrt{1+x^2}}}{1+x^2} = \frac{1}{(1+x^2)^{\frac{3}{2}}},$$

$$\sqrt{1-z^2} = \sqrt{1 - \frac{x^2}{1+x^2}} = \frac{1}{\sqrt{1+x^2}},$$

$$\therefore \frac{du}{dx} = \frac{1}{1+x^2}.$$

(7) $u = \log(x + \sqrt{1+x^2}) = \log z$.

$$\therefore \frac{du}{dx} = \frac{\frac{dz}{dx}}{z}, \text{ and } z = x + \sqrt{1+x^2};$$

$$\therefore \frac{dz}{dx} = 1 + \frac{x}{\sqrt{1+x^2}} = \frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}} = \frac{z}{\sqrt{1+x^2}};$$

$$\therefore \frac{du}{dx} = \frac{1}{\sqrt{1+x^2}}.$$

(8) $u = \log(\log x) = \log z$;

$$\therefore \frac{du}{dx} = \frac{du}{dz} \cdot \frac{dz}{dx} = \frac{1}{z} \cdot \frac{1}{x} = \frac{1}{x\log x}.$$

(9) $u = x^z$, where $z = f(x)$.

$$\log u = z \log x; \quad \therefore \frac{du}{dx} \cdot \frac{1}{u} = \frac{dz}{dx} \log x + z \cdot \frac{1}{x};$$

$$\therefore \frac{du}{dx} = x^z \left\{ \frac{z}{x} + \frac{dz}{dx} \cdot \log x \right\}.$$

(10) $u = z^v$, z and v being functions of x.

$$\log u = v \log z; \quad \therefore \frac{du}{dx} \cdot \frac{1}{u} = \frac{dv}{dx} \log z + v \cdot \frac{dz}{dx} \cdot \frac{1}{z};$$

$$\therefore \frac{du}{dx} = z^v \left\{ \frac{dv}{dx} \log z + \frac{v}{z} \cdot \frac{dz}{dx} \right\}.$$

(11) $u = e^{e^x} = e^z$, if $z = e^x$;

$$\therefore \frac{du}{dx} = e^z \cdot \frac{dz}{dx} = e^z \cdot e^x = e^{e^x} e^x.$$

(12) $u = z^{v^y}$, where z, v, and y are functions of x.

Let $v^y = v_1$; $\therefore u = z^{v_1}$,

$$\text{and } \frac{du}{dx} = z^{v_1} \left\{ \log z \cdot \frac{dv_1}{dx} + \frac{v_1}{z} \cdot \frac{dz}{dx} \right\}.$$

$$\text{But } \because v_1 = v^y; \quad \therefore \frac{dv_1}{dx} = v^y \left\{ \log v \cdot \frac{dy}{dx} + \frac{y}{v} \cdot \frac{dv}{dx} \right\};$$

$$\therefore \frac{du}{dx} = z^{v^y} \left\{ v^y \cdot \log z \left(\log v \cdot \frac{dy}{dx} + \frac{y}{v} \cdot \frac{dv}{dx} \right) + \frac{v^y}{z} \cdot \frac{dz}{dx} \right\}$$

$$= z^{v^y} \cdot v^y \left\{ \log z \cdot \log v \cdot \frac{dy}{dx} + \frac{y}{v} \log z \cdot \frac{dv}{dx} + \frac{1}{z} \cdot \frac{dz}{dx} \right\}.$$

(13) $u = \log (x + 1 + \sqrt{2x + x^2})$; $\therefore \dfrac{du}{dx} = \dfrac{1}{\sqrt{2x + x^2}}$.

(14) $u = \log \dfrac{\sqrt{x^2+1} - x}{\sqrt{x^2+1} + x}$; $\dfrac{du}{dx} = -\dfrac{2}{\sqrt{x^2+1}}$.

(15) $u = \log \dfrac{x}{\sqrt{x^2+1} + x}$; $\dfrac{du}{dx} = \dfrac{1}{x} - \dfrac{1}{\sqrt{x^2+1}}$.

(16) $= u \log \dfrac{x}{\sqrt{x^2+1} + 1}$; $\dfrac{du}{dx} = \dfrac{1}{x\sqrt{x^2+1}}$.

(17) $u = (\log x)^n$; $\therefore \dfrac{du}{dx} = n \cdot (\log x)^{n-1} \cdot \dfrac{1}{x}$.

(18) $u = x^x$; $\dfrac{du}{dx} = x^x \{1 + \log x\} = x^x \log (ex)$.

(19) $u = (\sin x)^{\cos x}$;

$$\therefore \frac{du}{dx} = \overline{\sin x}\rceil^{\cos x} \left\{ -\sin x \log \sin x + \frac{\cos^2 x}{\sin x} \right\}.$$

(20) $u = \log \tan x$; $\dfrac{du}{dx} = \dfrac{2}{\sin 2x}$.

(21) $u = \log \sqrt{\dfrac{1+\sin x}{1-\sin x}}$; $\dfrac{du}{dx} = \dfrac{1}{\cos x}$.

(22) $u = \log(\cos x + \sqrt{-1}\sin x)$; $\dfrac{du}{dx} = \sqrt{-1}$.

(23) $u = e^x(x^4 - 4x^3 + 12x^2 - 24x + 24)$; $\dfrac{du}{dx} = x^4 . e^x$.

(24) $u = \sin^3 x \cos x$; $\dfrac{du}{dx} = \sin^2 x(3 - 4\sin^2 x)$.

(25) $u = \dfrac{\cos x}{\cos 2x}$; $\dfrac{du}{dx} = \dfrac{\sin x(2 + \cos 2x)}{(\cos 2x)^2}$.

(26) $u = e^x . \cos x$; $\dfrac{du}{dx} = e^x(\cos x - \sin x)$.

(27) $u = \sin^{-1}(3x - 4x^3)$; $\dfrac{du}{dx} = \dfrac{3}{\sqrt{1-x^2}}$.

(28) $u = \cos^{-1}(4x^3 - 3x)$; $\dfrac{du}{dx} = \dfrac{-3}{\sqrt{1-x^2}}$.

(29) $u = \tan^{-1} . \dfrac{2x}{1-x^2}$; $\dfrac{du}{dx} = \dfrac{2}{1+x^2}$.

(30) $u = \dfrac{1}{\sqrt{a^2-b^2}} . \cos^{-1}\left(\dfrac{b + a\cos x}{a + b\cos x}\right)$;

$$\therefore \frac{du}{dx} = \frac{1}{a + b . \cos x}.$$

(31) $u = e^{\sin x} . \cos x$; $\dfrac{du}{dx} = e^{\sin x}(1 - \sin x - \sin^2 x)$.

(32) $u = \log \sqrt{\dfrac{1-\cos x}{1+\cos x}}$; $\dfrac{du}{dx} = \dfrac{1}{\sin x}$.

(33) $u = \sin^{-1}(2x - 1)$; $\dfrac{du}{dx} = \dfrac{1}{\sqrt{x - x^2}}$.

(34) $u = (\sin x)^x$; $\dfrac{du}{dx} = (\sin x)^x \{\log \sin x + x \cot x\}$.

(35) $u = \dfrac{x^4 (\log x)^2}{4} - \dfrac{x^4 \log x}{8} + \dfrac{x^4}{32}$; $\dfrac{du}{dx} = x^3 (\log x)^2$.

(36) $u = \tan^{-1}(\sqrt{1+x^2} - x)$; $\dfrac{du}{dx} = -\frac{1}{2} . \dfrac{1}{1+x^2}$.

(37) $u = \log \sqrt{\sin x} + \log \sqrt{\cos x}$; $\dfrac{du}{dx} = \cot 2x$.

(38) $u = \sqrt{1-x^2} + \sin^{-1} x$; $\dfrac{du}{dx} = \sqrt{\dfrac{1-x}{1+x}}$.

(39) $u = \dfrac{1}{\sqrt{2}} \sin^{-1} . \dfrac{x\sqrt{2}}{1+x^2}$; $\dfrac{du}{dx} = \dfrac{1-x^2}{1+x^2} . \dfrac{1}{\sqrt{1+x^4}}$.

(40) $u = \dfrac{2}{\sqrt{3}} \tan^{-1} . \left(\dfrac{2x+1}{\sqrt{3}}\right)$; $\dfrac{du}{dx} = \dfrac{1}{1+x+x^2}$.

(41) $u = \log \sqrt[3]{\dfrac{x^3-1}{x^3+1}}$; $\dfrac{du}{dx} = \dfrac{2x^2}{x^6-1}$.

(42) $u = x^{\frac{1}{x}}$; $\dfrac{du}{dx} = \dfrac{x^{\frac{1}{x}}}{x^2} \log\left(\dfrac{e}{x}\right)$.

(43) $u = \log \dfrac{(x+2)^2}{\sqrt{x+1} . (x+3)^{\frac{3}{2}}}$; $\dfrac{du}{dx} = \dfrac{x}{x^3 + 6x^2 + 11x + 6}$.

(44) $u = \log \sqrt[4]{\dfrac{1+x}{1-x}} + \frac{1}{2} \tan^{-1} x$; $\dfrac{du}{dx} = \dfrac{1}{1-x^4}$.

(45) $u = \dfrac{e^{ax}(a \sin x - \cos x)}{a^2+1}$; $\dfrac{du}{dx}\, e^{ax} . \sin x$.

(46) $u = a \log\left(\dfrac{a + \sqrt{a^2 - x^2}}{x}\right) - \sqrt{a^2 - x^2}$; $\dfrac{du}{dx} = \dfrac{-\sqrt{a^2-x^2}}{x}$.

(47) $u = a(\sin x - \cos x)$; $\dfrac{du}{dx} = \sqrt{2a^2 - u^2}$.

(48) $u = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$; $\dfrac{du}{dx} = 1 - u^2$.

(49) $u = xe^{\tan^{-1} x}$; $\dfrac{du}{dx} = e^{\tan^{-1} x}\left(\dfrac{1+x+x^2}{1+x^2}\right)$.

(50) $u = \sin(\log x)$; $\dfrac{du}{dx} = \dfrac{1}{x} \cos(\log x)$.

(51) $u = a(z + \sin z)$; $x = a(1 - \cos z)$;

$$\frac{du}{dx} = \sqrt{\frac{2ax - x^2}{x}}.$$

(52) $u = e^x \sqrt{\dfrac{1+x}{1-x}}$; $\dfrac{du}{dx} = e^x \cdot \dfrac{2 - x^2}{(1-x)^{\frac{3}{2}}\sqrt{1+x}}$.

(53) $u = -\frac{1}{3}\dfrac{\sec x}{(\sin x)^3} - \frac{8}{3}\cot 2x$; $\dfrac{du}{dx} = \dfrac{1}{(\sin x)^4 (\cos x)^2}$.

(54) $u = \sin^{-1}(2x\sqrt{1 - x^2})$; $\dfrac{du}{dx} = \dfrac{2}{\sqrt{1 - x^2}}$.

(55) $u = 2e^{\sqrt{x}}(x^{\frac{3}{2}} - 3x + 6\sqrt{x} - 6)$; $\therefore \dfrac{du}{dx} = xe^{\sqrt{x}}$.

(56) $u = x^{x^n}$; $\dfrac{du}{dx} = x^{x^n} x^{n-1} \log(ex^n)$.

(57) $u = x^{n^x}$; $\dfrac{du}{dx} = x^{n^x} n^x \left\{\dfrac{1}{x} + \log x \,.\, \log n\right\}$.

(58) $u = n\dfrac{x^{\frac{1}{n}}}{\sqrt{1 - x^{\frac{2}{n}}}} - n \sin^{-1} . \, x^{\frac{1}{n}}$;

$$\frac{du}{dx} = \frac{x^{\frac{3}{n}}}{x\left(1 - x^{\frac{2}{n}}\right)^{\frac{3}{2}}}.$$

(59) If $u\sqrt{1 - x^2} + \sqrt{1 - u^2} = a^2$.

$$\frac{du}{dx} + \frac{\sqrt{1 - u^2}}{\sqrt{1 - x^2}} = 0.$$

(60) If $\sqrt{1 - x^2} + \sqrt{1 - u^2} = a(x - u)$.

$$\frac{du}{dx} = \frac{\sqrt{1 - u^2}}{\sqrt{1 - x^2}}.$$

CHAPTER III.

Successive Differentiation. Maclaurin's Theorem.

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51. If $u=f(x)$; $\frac{du}{dx}$, or as it is frequently called, its derivative may also be a function of $x, f'(x)$, and is capable of being differentiated.

For suppose that $\frac{du}{dx}=p$, and for x put $x+h$,

then $\frac{du}{dx}+\frac{d.\left(\frac{du}{dx}\right)}{dx}h+\&c.=p+\frac{dp}{dx}h+\&c.$ (by writing $\frac{du}{dx}$ for u, in $u+\frac{du}{dx}h+Uh^2$);

$$\therefore \frac{d.\left(\frac{du}{dx}\right)}{dx}=\frac{dp}{dx}.$$

So also $\frac{dp}{dx}$ may be a function of x, equal q,

$$\text{and } \frac{dq}{dx}=\frac{d.\left(\frac{dp}{dx}\right)}{dx}=\frac{d}{dx}.\left\{\frac{d.\left(\frac{du}{dx}\right)}{dx}\right\};$$

and so on for other differential coefficients.

This process is called successive differentiation, and $\frac{du}{dx}$, $\frac{dp}{dx}$, $\frac{dq}{dx}$, &c. are called the first, second, third, &c. differential coefficients, or the first, second, third, &c. derivatives of u.

A more convenient notation than that above is used, the reason for which may be derived from the consideration of differentials; and here we may remark that dx as well as h, is always considered to be invariable, when $u=f(x)$.

Now since $\frac{du}{dx}=p$; $\therefore du=pdx$;

$$\therefore d(du)=dpdx: \text{ but } dp=qdx;$$

$$\therefore d(du)=qdx^2.$$

But because $d(du)$ is the symbol for shewing that u has been twice differentiated, and since d is the symbol of differentiation; therefore d^2u will fitly express the fact of u being thus twice operated upon;

$$\therefore\ d^2u = q\,dx^2.$$

Similarly
$$d(d^2u) = d^3u = dq\,dx^2 = r\,dx^3,$$
$$d(d^3u) = d^4u = dr\,dx^3 = s\,dx^4,$$
&c.;

$$\therefore\ \frac{dp}{dx} = q = \frac{d^2u}{dx^2},$$
$$\frac{dq}{dx} = r = \frac{d^3u}{dx^3},$$
$$\frac{dr}{dx} = s = \frac{d^4u}{dx^4};$$

and the n^{th} differential coefficient is $\dfrac{d^nu}{dx^n}$.

Observe $\dfrac{d^2u}{dx^2}$, $\dfrac{d^3u}{dx^3}$, &c. are most commonly read, second *du* by *dx* squared, third *du* by *dx* cubed; but sometimes, *d* two *u*, by *dx* squared, *d* three *u*, by *dx* cubed, &c.

Ex. 1. Let $u = x^4 + x^3 + x^2 + x + 1$,

$$\frac{du}{dx} = 4x^3 + 3x^2 + 2x + 1,$$
$$\frac{d^2u}{dx^2} = 3 \,.\, 4x^2 + 2 \,.\, 3x + 2,$$
$$\frac{d^3u}{dx^3} = 2 \,.\, 3 \,.\, 4x + 2 \,.\, 3,$$
$$\frac{d^4u}{dx^4} = 2 \,.\, 3 \,.\, 4,$$
$$\frac{d^5u}{dx^5} = 0.$$

Ex. 2. Let $u = \dfrac{1}{x} = x^{-1}$.

$$\frac{du}{dx} = -x^{-2} = -\frac{1}{x^2},$$
$$\frac{d^2u}{dx^2} = 2x^{-3} = \frac{2}{x^3},$$
$$\frac{d^3u}{dx^3} = -2 \,.\, 3x^{-4} = -\frac{2 \,.\, 3}{x^4},$$

$$\frac{d^4u}{dx^4} = 2 \, . \, 3 \, . \, 4x^{-5} = \frac{2 \, . \, 3 \, . \, 4}{x^5},$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\frac{d^nu}{dx^n} = (-1)^n 2 \, . \, 3 \, . \, 4 \, . \, 5 \ldots n \, . \, x^{-(n+1)}.$$

Ex. 3. Let $u = \sin(ax + b)$; find $\frac{d^4u}{dx^4}$.

$$\therefore \frac{du}{dx} = a \cos(ax + b),$$

$$\frac{d^2u}{dx^2} = -a^2 \sin(ax + b),$$

$$\frac{d^3u}{dx^3} = -a^3 \cos(ax + b),$$

$$\frac{d^4u}{dx^4} = a^4 \sin(ax + b) = a^4u.$$

Ex. 4. Let $u = e^{ax}$; find $\frac{d^nu}{dx^n}$.

$$\therefore \frac{du}{dx} = ae^{ax}; \quad \frac{d^2u}{dx^2} = a^2e^{ax} = a^2u,$$

$$\frac{d^3u}{dx^3} = a^2 \frac{du}{dx} = a^3e^{ax} = a^3u,$$

$$\frac{d^4u}{dx^4} = a^3 \frac{du}{dx} = a^4e^{ax} = a^4u;$$

$$\therefore \frac{d^nu}{dx^n} = a^ne^{ax} = a^nu.$$

Ex. 5. Let $u = zv$; to find d^2u, d^3u, &c.

$$\because du = zdv + vdz \ldots\ldots\ldots (1);$$

$$\therefore d^2u = d(zdv) + d(vdz).$$

But from (1), $d(zdv) = zd^2v + dzdv$,

$$d(vdz) = vd^2z + dvdz;$$

$$\therefore d^2u = zd^2v + 2dzdv + vd^2z;$$

$$\therefore d^3u = d(zd^2v) + 2 \, . \, d(dzdv) + d(vd^2z).$$

But $d(z \, . \, d^2v) = zd^3v + dzd^2v$,

$$2d(dzdv) = 2d^2zdv + 2dzd^2v,$$

$$d \, . \, (vd^2z) = vd^3z + dvd^2z;$$

$$\therefore d^3u = zd^3v + 3dzd^2v + 3dvd^2z + vd^3z,$$

and $d^4u = zd^4v + 4dzd^3v + 6d^2zd^2v + 4d^3zdv + vd^4z.$

Since the law of the numerical coefficients is apparently that of the coefficients of $(a+b)^n$;

$$\therefore\ d^n u = z d^n v + n \,.\, dz d^{n-1} v + n \,.\, \frac{(n-1)}{2} d^2 z d^{n-2} v + \&c.;$$

$$\therefore\ \frac{d^n u}{dx^n} = z \,.\, \frac{d^n v}{dx^n} + n \frac{dz}{dx} \,.\, \frac{d^{n-1} v}{dx^{n-1}} + n \,.\, \frac{(n-1)}{2} \,.\, \frac{d^2 z}{dx^2} \,.\, \frac{d^{n-2} v}{dx^{n-2}} + \&c.$$

a theorem due to Leibnitz, and which may be used to find the differential coefficient of the product of two functions.

To prove the law of the coefficients. Let

$$d^n u = z d^n v + n dz d^{n-1} v + n \frac{n-1}{2} \,.\, d^2 z d^{n-2} v + \&c.;$$

$$\therefore\ d^{n+1} u = z d^{n+1} v + dz d^n v + n \,.\, (dz d^n v + d^2 z d^{n-1} v)$$

$$+ n \frac{n-1}{2} \,.\, (d^2 z d^{n-1} v + d^3 z \,.\, d^{n-2} v) + \&c.$$

$$= z d^{n+1} v + (n+1) \,.\, dz d^n v + \frac{n \,.\, (n+1)}{2} \,.\, d^2 z \,.\, d^{n-1} v + \&c.,$$

which shews that if the theorem be true for n, it is true for $n+1$, and it has been shewn to be true for $n=2$, and $n=3$; it is $\therefore$ true when $n=4$; and $\therefore$ when n is any integer.

Expansion of Functions.

52. If $u = f(x)$ can be expanded into a series of the form

$$u = A + Bx + Cx^2 + Dx^3 + Ex^4 \ \&c.$$

where A, B, C, &c. are constant, to find these coefficients.

This is Maclaurin's or Stirling's Theorem.

Since $u = A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c.$

By successive differentiation

$$\frac{du}{dx} = B + 2Cx + 3Dx^2 + 4E \,.\, x^3 + \&c.$$

$$\frac{d^2 u}{dx^2} = 2C + 2 \,.\, 3Dx + 3 \,.\, 4E \,.\, x^2 + \&c.$$

$$\frac{d^3 u}{dx^3} = 2 \,.\, 3 \,.\, D + 2 \,.\, 3 \,.\, 4E \,.\, x + \&c.$$

$$\frac{d^4 u}{dx^4} = 2 \,.\, 3 \,.\, 4E + \&c.$$

&c. = &c.

Make $x=0$, and let U_0, U_1, U_2, U_3, &c. represent the resulting values of u, $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$, &c.

$$\therefore\ U_0=A;\ U_1=B;\ U_2=2C;\ \therefore\ C=\frac{U_2}{1\,.\,2},$$

$$U_3=2\,.\,3D;\ \therefore\ D=U_3\frac{1}{2\,.\,3},$$

$$E=U_4\,.\,\frac{1}{2\,.\,3\,.\,4},\ \text{\&c.}=\text{\&c.};$$

$$\therefore\ u=U_0+U_1x+U_2\,.\,\frac{x^2}{1\,.\,2}+U_3\,.\,\frac{x^3}{2\,.\,3}+U_4\,.\,\frac{x^4}{2\,.\,3\,.\,4}+\text{\&c.}$$

COR. The general term is obviously $U_n\frac{x^n}{1\,.\,2\,.\,3\ldots n}$.

Examples of the Expansion of Functions.

(1) Let $u=(x+a)^4$; $\qquad\therefore\ U_0=a^4$; if $x=0$;

$$\frac{du}{dx}=4(x+a)^3;\qquad\therefore\ U_1=4a^3,$$

$$\frac{d^2u}{dx^2}=4\,.\,3(x+a)^2;\qquad\therefore\ U_2=3\,.\,4a^2,$$

$$\frac{d^3u}{dx^3}=2\,.\,3\,.\,4(x+a);\ \therefore\ U_3=2\,.\,3\,.\,4a,$$

$$\frac{d^4u}{dx^4}=2\,.\,3\,.\,4;\qquad\therefore\ U_4=2\,.\,3\,.\,4,$$

$$\frac{d^5u}{dx^5}=0;\ \therefore\ U_5=0,\ \text{and}\ U_6,\ U_7,\ \text{\&c. each}=0;$$

$$\therefore\ u=(x+a)^4=a^4+4a^3x+\frac{3\,.\,4}{1\,.\,2}a^2x^2+\frac{2\,.\,3\,.\,4}{1\,.\,2\,.\,3}ax^3+\frac{2\,.\,3\,.\,4}{2\,.\,3\,.\,4}x^4$$

$$=a^4+4a^3x+6a^2x^2+4ax^3+x^4.$$

(2) Expand $(a+bx+cx^2)^n$.

$$u=(a+bx+cx^2)^n;\ \therefore\ U_0=a^n,$$

$$\frac{du}{dx}=n\,.\,(a+bx+cx^2)^{n-1}(b+2cx);\ \therefore\ U_1=nba^{n-1},$$

$$\frac{d^2u}{dx^2}=n(n-1)(a+bx+cx^2)^{n-2}(b+2cx)^2+2cn\,.\,(a+bx+cx^2)^{n-1};$$

$$\therefore\ U_2 = n . (n-1)\, a^{n-2} b^2 + n . 2c . a^{n-1},$$

$$\frac{d^3u}{dx^3} = n . (n-1) . (n-2)\,(a+bx+cx^2)^{n-3}\,(b+2cx)^3$$
$$+ 2n . (n-1)\,(a+bx+cx^2)^{n-2}\, 2c . (b+2cx)$$
$$+ 2c . n\,(n-1) . (a+bx+cx^2)^{n-2}\,(b+2cx);$$

$$\therefore\ U_3 = n(n-1)(n-2) . a^{n-3} b^3 + 2 . 3n . (n-1)\, a^{n-2} bc,$$

&c. = &c.

$$\therefore\ (a+bx+cx^2)^n = a^n + na^{n-1}\, bx + \{n . \frac{(n-1)}{2} . a^{n-2} b^2 + na^{n-1} c\} x^2$$
$$+ \left\{ \frac{n . (n-1)(n-2)}{1 . 2 . 3} a^{n-3} b^3 + n . (n-1) . a^{n-2} bc \right\} x^3$$
$$+ \&c.$$

(3) Expand $\sin x$ and $\cos x$ in terms of x.

If $u = \sin x$,	If $u = \cos x$,
$\frac{du}{dx} = \cos x$,	$\frac{du}{dx} = -\sin x$,
$\frac{d^2u}{dx^2} = -\sin x$,	$\frac{d^2u}{dx^2} = -\cos x$,
$\frac{d^3u}{dx^3} = -\cos x$,	$\frac{d^3u}{dx^3} = \sin x$,
$\frac{d^4u}{dx^4} = +\sin x$,	$\frac{d^4u}{dx^4} = \cos x$,
&c. = &c.	&c. = &c.

After the 4th differentiation the values recur.

Make $x = 0$, then in the series for $\sin x$,

$$U_0 = 0,\quad U_1 = 1,\quad U_2 = 0,\quad U_3 = -1,\quad U_4 = 0,\quad U_5 = 1,\ \&c.,$$

and in the series for $\cos x$,

$$U_0 = 1,\quad U_1 = 0,\quad U_2 = -1,\quad U_3 = 0,\quad U_4 = 1,\ \&c.$$

$$\therefore\ \sin x = x - \frac{x^3}{2 . 3} + \frac{x^5}{2 . 3 . 4 . 5} - \&c.$$

$$\text{and } \cos x = 1 - \frac{x^2}{1 . 2} + \frac{x^4}{2 . 3 . 4} - \&c.$$

Cor. The series for $\cos x$ may be derived from that of $\sin x$ by differentiating the latter.

(4) Similarly, if $u = \tan x$, we may find $\tan x$ in terms of x; but more readily in the following manner*.

Let $u = \tan x = a_1 x + a_3 x^3 + a_5 x^5 + a_7 x^7 + \&c.$;

$$\therefore \frac{du}{dx} = 1 + \tan^2 x = a_1 + 3a_3 x^2 + 5a_5 x^4 + 7a_7 x^6 + \&c.$$

But $1 + \tan^2 x = 1 + (a_1 x + a_3 x^3 + a_5 x^5 + \&c.)^2$

$$= 1 + a_1^2 x^2 + 2a_1 a_3 x^4 + (a_3^2 + 2a_1 a_5)\, x^6 + \&c.;$$

therefore equating coefficients of the like powers of x,

$$a_1 = 1, \quad 3a_3 = a_1{}^2; \quad \therefore a_3 = \frac{1}{3},$$

$$5a_5 = 2a_1 a_3 = \frac{2}{3}; \quad \therefore a_5 = \frac{2}{3.5},$$

$$7a_7 = a_3{}^2 + 2a_1 a_5 = \frac{1}{9} + \frac{4}{3.5} = \frac{17}{5.9}; \quad \therefore a_7 = \frac{17}{5.7.9};$$

$$\therefore u = x + \frac{x^3}{1.3} + \frac{2x^5}{3.5} + \frac{17x^7}{5.7.9} + \&c.$$

(5). In a similar manner may $\sin x$ be found.

Let $\sin x = a_1 x + a_3 x^3 + a_5 x^5 + \&c.$;

$\therefore \cos x = a_1 + 3a_3 x^2 + 5a_5 x^4 + \&c.$;

$\therefore -\sin x = 2.3a_3 x + 4.5a_5 x^3 + \&c.$;

$$= -a_1 x - a_3 x^3 - a_5 x^5 - \&c.;$$

$$\therefore a_3 = -\frac{a_1}{2.3}; \quad a_5 = -\frac{a_3}{4.5} = \frac{a_1}{2.3.4.5};$$

$$\therefore \sin x = a_1\left(x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \&c.\right);$$

$$\therefore \frac{\sin x}{x} = a_1\left(1 - \frac{x^2}{2.3} + \&c.\right),$$

$$\text{if } x = 0; \quad \frac{\sin x}{x} = 1; \quad \therefore a_1 = 1;$$

$$\therefore \sin x = x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \&c.$$

* That $\tan x$ can only involve odd powers of (x) may be thus shewn:

Let $\tan x = a_1 x + b_2 x^2 + a_3 x^3 + b_4 x^4 + a_5 x^5 + \&c.$;

$\therefore \tan(-x) = -a_1 x + b_2 x^2 - a_3 x^3 + b_4 x^4 - a_5 x^5 + \&c.$;

$\therefore \tan x - \tan(-x) = 2a_1 x + 2a_3 x^3 + 2a_5 x^5 + \&c.$

But $\tan(-x) = -\tan(x)$; $\therefore \tan x - \tan(-x) = 2\tan x$;

$\therefore \tan x = a_1 x + a_3 x^3 + a_5 x^5 + \&c.$

(6) $u = \sin^{-1}x$, whence if $x = 0$, $U_0 = \sin^{-1}0 = 0$,

and $\dfrac{du}{dx} = \dfrac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} = 1 + \dfrac{1}{2}x^2 + \dfrac{1.3}{2.4}x^4 + \dfrac{1.3.5}{2.4.6}x^6 + \&c.$

but from Maclaurin's Theorem,

$$\frac{du}{dx} = U_1 + 2U_2 . \frac{x}{1.2} + 3U_3 . \frac{x^2}{2.3} + 4U_4 . \frac{x^3}{2.3.4} + 5U_5 . \frac{x^4}{2.3.4.5} + \&c.$$

$$\therefore\ U_1 = 1,\quad U_2 = 0,\quad \frac{3U_3}{2.3} = \frac{1}{2};\quad \therefore\ U_3 = 1,\quad U_4 = 0,$$

$$\frac{5U_5}{2.3.4.5} = \frac{1.3}{2.4},\quad U_5 = 1.3^2,\quad U_6 = 0,$$

$$\frac{7U_7}{2.3.4.5.6.7} = \frac{1.3.5}{2.4.6};\quad \therefore\ U_7 = 1^2.3^2.5^2;$$

$$\therefore\ \sin^{-1}x = x + \frac{x^3}{1.2.3} + \frac{1.3^2x^5}{1.2.3.4.5} + \frac{1^2.3^2.5^2.x^7}{1.2.3.4.5.6.7} + \&c.$$

$$= x + \frac{1}{1.2}.\frac{x^3}{3} + \frac{1.3}{2.4}.\frac{x^5}{5} + \frac{1.3.5}{2.4.6}.\frac{x^7}{7} + \&c.$$

the general term of this series (Newton's) is obviously

$$\frac{1.3.5 \ldots\ldots (2n-3)}{2.4.6 \ldots\ldots (2n-2)} . \frac{x^{2n-1}}{2n-1}.$$

By this series, the length of a circular arc may be found; thus, let $\sin^{-1}x = 30$; $\therefore\ x = \dfrac{1}{2}$; and the length of 30°,

$$= \frac{1}{2} + \frac{1}{1.2}.\frac{1}{3 \times 8} + \frac{1.3}{2.4}.\frac{1}{32 \times 5} + \&c.$$

(7) The same series may be also obtained, thus,

let $u = \sin^{-1}x = a_1x + a_3x^3 + a_5x^5 + a_7x^7 + \&c.$;

for $\sin^{-1}x$ cannot contain even powers of x;

$$\therefore\ \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} = a_1 + 3a_3x^2 + 5a_5x^4 + 7a_7x^6 + \&c.$$

$$\text{But } \frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{1.3}{2.4}.x^4 + \frac{1.3.5}{2.4.6}x^6 + \&c.;$$

$$\therefore\ a_1 = 1,\quad 3a_3 = \frac{1}{2};\quad \therefore\ a_3 = \frac{1}{2}.\frac{1}{3},$$

$$5a_5 = \frac{1.3}{2.4};\quad \therefore\ a_5 = \frac{1.3}{2.4}.\frac{1}{5},$$

$$7a_7 = \frac{1.3.5}{2.4.6};\quad \therefore\ a_7 = \frac{1.3.5}{2.4.6}.\frac{1}{7};$$

$$\therefore \sin^{-1}x = \frac{x}{1} + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7} + \&c.$$

(8) $u = \tan^{-1}x$; $x = 0$, $U_0 = \tan^{-1}0 = 0$;

$$\therefore \frac{du}{dx} = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \&c.$$

$$= U_1 + 2U_2\frac{x}{2} + \frac{3U_3x^2}{1.2.3} + \frac{4U_4x^3}{2.3.4} + \frac{5U_5x^4}{2.3.4.5} + \&c.$$

$$\therefore U_1 = 1,\quad U_2 = 0,\quad \frac{U_3}{2} = -1;\quad \therefore U_3 = -2,\quad U_4 = 0,$$

$$\frac{U_5}{2.3.4} = 1;\quad \therefore U_5 = 2.3.4;$$

$$\therefore u = x - \frac{2x^3}{2.3} + \frac{2.3.4x^5}{2.3.4.5} - \&c.$$

$$\text{or } \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \&c.$$

Gregory's series for the arc in terms of the tangent.

53. Hence may be found approximate expressions for the length of the arc of a circle.

(1°) Let $\tan^{-1}x = \frac{\pi}{4}$; $\therefore x = \tan\frac{\pi}{4} = 1$,

$$\text{and } \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \&c.$$

(2°) Since $\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$,

$$\text{and } \tan^{-1}\frac{1}{2} = \frac{1}{2} - \frac{1}{3}\cdot\frac{1}{2^3} + \frac{1}{5.2^5} - \&c.;$$

$$\tan^{-1}\frac{1}{3} = \frac{1}{3} - \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{5.3^5} - \&c.;$$

$$\therefore \frac{\pi}{4} = \left(\frac{1}{2} + \frac{1}{3}\right) - \frac{1}{3}\cdot\left(\frac{1}{2^3} + \frac{1}{3^3}\right) + \frac{1}{5}\cdot\left(\frac{1}{2^5} + \frac{1}{3^5}\right) - \&c.$$

(3°) A very convergent series is given by Machin's formula,

$$\text{that } \frac{\pi}{4} = 4\tan^{-1}.\frac{1}{5} - \tan^{-1}.\frac{1}{239}.$$

To prove this, let $A = 4\tan^{-1}\frac{1}{5} = 4a$.

$$\text{Then } \tan A = \frac{4\tan a - 4\tan^3 a}{1 - 6\tan^2 a + \tan^4 a} = \frac{\frac{4}{5} - \frac{4}{125}}{1 - \frac{6}{25} + \frac{1}{625}}$$

$$= \frac{4(125-5)}{625 - 150 + 1} = \frac{4 \times 120}{476} = \frac{120}{119}; \quad \therefore > 1,$$

$$\text{and } \tan(A - 45^0) = \frac{\tan A - 1}{\tan A + 1} = \frac{\frac{120}{119} - 1}{\frac{120}{119} + 1} = \frac{1}{239};$$

$$\therefore A - 45^0 = \tan^{-1}\frac{1}{239};$$

$$\therefore 45^0 = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$$

$$= 4\left\{\frac{1}{5} - \frac{1}{3}\cdot\frac{1}{5^3} + \frac{1}{5}\cdot\frac{1}{5^5} - \&c.\right\}$$

$$- \left\{\frac{1}{239} - \frac{1}{3}\cdot\frac{1}{(239)^3} + \frac{1}{5}\cdot\frac{1}{(239)^5} - \&c.\right\}.$$

54. Log x cannot be found by Maclaurin's Theorem, since U_0, U_1, U_2, &c. are infinite: but log $(1+x)$ may be.

Suppose the logarithms to be Napierian, where $A = 1$.

$$u = \log(1+x); \quad \therefore U_0 = \log(1) = 0,$$

$$\frac{du}{dx} = \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \&c.$$

$$\text{But } \frac{du}{dx} = U_1 + U_2 x + \frac{U_3 x^2}{2} + \frac{U_4 x^3}{2.3} + \frac{U_5 x^4}{2.3.4} + \&c.$$

$$\therefore U_1 = 1, \quad U_2 = -1, \quad U_3 = 2, \quad U_4 = -2.3, \quad U_5 = 2.3.4;$$

$$\therefore u = \log(1+x) = x - \frac{x^2}{2} + \frac{2x^3}{2.3} - \frac{2.3x^4}{2.3.4} + \frac{2.3.4x^5}{2.3.4.5} + \&c.$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \&c.$$

Cor. Had a been the base, then (Art. 48),

$\dfrac{du}{dx} = \dfrac{1}{A}\cdot\dfrac{1}{x+1}$, and if $\log_a$ be a log in that system,

$$\log_a(1+x) = \frac{1}{A}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \&c.\right)$$
$$= \frac{1}{A}\log_e(1+x),$$

where $A = \log_e a$, the factor $\frac{1}{A}$ is called the modulus.

The series just obtained does not converge, but from it converging series may be derived. See Algebra, p. 270.

55. Expand a^x in ascending powers of x.

$$u = a^x, \quad x = 0; \quad \therefore U_0 = 1;$$
$$\frac{du}{dx} = A \cdot a^x \;\ldots\ldots\ldots\ldots\; \therefore U_1 = A,$$
$$\frac{d^2u}{dx^2} = A^2 a^x \;\ldots\ldots\ldots\ldots\; \therefore U_2 = A^2,$$
$$\frac{d^3u}{dx^3} = A^3 a^x \;\ldots\ldots\ldots\ldots\; \therefore U_3 = A^3,$$
$$\vdots$$
$$\frac{d^nu}{dx^n} = A^n a^x \;\ldots\ldots\ldots\ldots\; \therefore U_n = A^n;$$
$$\therefore a^x = 1 + Ax + \frac{A^2x^2}{1.2} + \frac{A^3x^3}{2.3} + \frac{A^4x^4}{2.3.4} + \&c.$$

COR. 1. If $a = e$; $A = \log_e e = 1$; and

$$e^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \frac{x^5}{2.3.4.5} + \&c.$$

56. In the expansion for e^x,

put successively for x, $x\sqrt{-1}$, and $-x\sqrt{-1}$;

$$\therefore e^{x\sqrt{-1}} = 1 + x\sqrt{-1} - \frac{x^2}{2} - \frac{x^3\sqrt{-1}}{2.3} + \frac{x^4}{2.3.4} + \frac{x^5\sqrt{-1}}{2.3.4.5} - \&c.$$
$$e^{-x\sqrt{-1}} = 1 - x\sqrt{-1} - \frac{x^2}{2} + \frac{x^3\sqrt{-1}}{2.3} + \frac{x^4}{2.3.4} - \frac{x^5\sqrt{-1}}{2.3.4.5} - \&c.$$

Therefore first by addition and then by subtraction,

$$e^{x\sqrt{-1}} + e^{-x\sqrt{-1}} = 2\left\{1 - \frac{x^2}{2} + \frac{x^4}{2.3.4} - \&c.\right\} = 2\cos x. \quad (1).$$
$$e^{x\sqrt{-1}} - e^{-x\sqrt{-1}} = 2\sqrt{-1}\left\{x - \frac{x^3}{2.3} + \&c.\right\} = 2\sqrt{-1}\sin x. \quad (2).$$

Again, adding and dividing by 2,

$$e^{x\sqrt{-1}} = \cos x + \sqrt{-1}\sin x.$$

Also by subtraction and dividing by 2,

$$e^{-x\sqrt{-1}} = \cos x - \sqrt{-1}\sin x.$$

Cor. 1. Hence $\cos x = \dfrac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2}$,

and $\sin x = \dfrac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}}$;

$$\therefore \tan x = \frac{1}{\sqrt{-1}}\,\frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}} = \frac{1}{\sqrt{-1}}\left(\frac{e^{2x\sqrt{-1}} - 1}{e^{2x\sqrt{-1}} + 1}\right).$$

Cor. 2. These equations have been proved independently of the value of x, we may therefore put mx for x;

$$\therefore \cos mx + \sqrt{-1}\sin mx = e^{mx\sqrt{-1}} = \overline{e^{x\sqrt{-1}}}\big|^{m}$$
$$= (\cos x + \sqrt{-1}\sin x)^m,$$

the formula of De Moivre.

57. $\because \log(1+u) = u - \dfrac{u^2}{2} + \dfrac{u^3}{3} - \dfrac{u^4}{4} + \&\text{c.};$

$$\therefore \log\left(1 + \frac{1}{u}\right) = u^{-1} - \frac{u^{-2}}{2} + \frac{u^{-3}}{3} - \frac{u^{-4}}{4} + \&\text{c.};$$

$$\therefore \log\left\{\left(\frac{1+u}{1+\frac{1}{u}}\right)\right\} = \log u = (u - u^{-1}) - \tfrac{1}{2}(u^2 - u^{-2}) + \&\text{c.}$$

For u write $e^{x\sqrt{-1}}$; $\therefore \log u = x\sqrt{-1}$;

$$\therefore x\sqrt{-1} = (e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}) - \tfrac{1}{2}(e^{2x\sqrt{-1}} - e^{-2x\sqrt{-1}}) + \&\text{c.}$$
$$= 2\sqrt{-1}\,\{\sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \&\text{c.}\};$$
$$\therefore \frac{x}{2} = \sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \&\text{c.};$$
$$\therefore \tfrac{1}{2} = \cos x - \cos 2x + \cos 3x - \&\text{c., differentiating.}$$

58. By division, $\dfrac{e^{x\sqrt{-1}}}{e^{-x\sqrt{-1}}} = e^{2x\sqrt{-1}} = \dfrac{\cos x + \sqrt{-1}\sin x}{\cos x - \sqrt{-1}\sin x}$

$$= \frac{1 + \sqrt{-1}\tan x}{1 - \sqrt{-1}\tan x}.$$

$$2x\sqrt{-1} = \log(1 + \sqrt{-1}\tan x) - \log(1 - \sqrt{-1}\tan x).$$

But $\log(1+u) - \log(1-u) = 2\left\{u + \dfrac{u^3}{3} + \dfrac{u^5}{5} + \&\text{c.}\right\};$

$$\therefore\ 2x\sqrt{-1} = 2\{\sqrt{-1}\tan x + \tfrac{1}{3}(\sqrt{-1}\tan x)^3 + (\sqrt{-1}\tan x)^5 + \&c.\}$$

$$= 2\sqrt{-1}\{\tan x - \tfrac{1}{3}\tan^3 x + \tfrac{1}{5}\tan^5 x - \&c.\};$$

$$\therefore\ x = \tan x - \tfrac{1}{3}\tan^3 x + \tfrac{1}{5}\tan^5 x - \&c.,\ \text{(Ex. 8.)}$$

59. Let $u^2 - ux - a^2 = 0$; find u in terms of x.

If $x = 0$; u becomes U_0; $\therefore\ (U_0)^2 - a^2 = 0$; $U_0 = \pm a$.

Differentiating, $2u\dfrac{du}{dx} - x\dfrac{du}{dx} - u = 0$;

make $x = 0$; $\therefore\ u = \pm a$; $\therefore\ 2aU_1 - a = 0$; $U_1 = \frac{1}{2}$.

Differentiating a second time,

$$(2u - x)\frac{d^2u}{dx^2} + 2\frac{du^2}{dx^2} - 2\frac{du}{dx} = 0;$$

$$\therefore\ \pm 2aU_2 + 2\,.\,\frac{1}{4} - 1 = 0;\quad \therefore\ U_2 = \pm\frac{1}{4a};$$

$$\therefore\ (2u - x)\frac{d^3u}{dx^3} + \left(2\frac{du}{dx} - 1\right)\frac{d^2u}{dx^2} + \left(4\frac{du}{dx} - 2\right)\frac{d^2u}{dx^2} = 0;$$

$$\text{or } (2u - x)\frac{d^3u}{dx^3} + 3\left(2\frac{du}{dx} - 1\right)\frac{d^2u}{dx^2} = 0;$$

$$\text{if } x = 0;\ \because\ 2U_1 - 1 = 0;\ \therefore\ U_3 = 0;$$

$$\therefore\ u = \pm a + \tfrac{1}{2}x \pm \frac{1}{4a}\,\frac{x^2}{1\,.\,2} + \&c.$$

Examples.

(1) If $u = \dfrac{1}{1 + x^2}$; $\dfrac{d^4u}{dx^4} = 24\,.\left\{\dfrac{1 - 10x^2 + 5x^4}{(1 + x^2)^5}\right\}$.

(2) If $u = \dfrac{x}{\sqrt{1 - x^2}}$; $\dfrac{d^3u}{dx^3} = \dfrac{3\,.\,(1 + 4x^2)}{(1 - x^2)^{\frac{7}{2}}}$.

(3) If $u = \sin x$; $\dfrac{d^nu}{dx^n} = \sin\left(x + n\dfrac{\pi}{2}\right)$.

For $\dfrac{du}{dx} = \cos x = \sin\left(\dfrac{\pi}{2} - x\right) = \sin\left(x + \dfrac{\pi}{2}\right)$;

$$\frac{d^2u}{dx^2} = \cos\left(x + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2} + x + \frac{\pi}{2}\right) = \sin\left(x + 2\frac{\pi}{2}\right),$$

(4) If $u = \cos x$; $\dfrac{d^n u}{dx^n} = \cos\left(x + n\dfrac{\pi}{2}\right)$.

(5) If $u = x^n . e^x$; $\dfrac{d^4 u}{dx^4} = e^x \{x^n + 4nx^{n-1} + 6n(n-1)x^{n-2}$
$+ 4n.(n-1).(n-2)x^{n-3} + n(n-1)(n-2)(n-3).x^{n-4}\}$.

(6) If $u = e^x \sin x$, $\dfrac{d^2 u}{dx^2} = 2e^x \cos x$,

$$\frac{d^4 u}{dx^4} = -4u, \quad \frac{d^8 u}{dx^8} = 4^2 u, \quad \frac{d^{12} u}{dx^{12}} = -4^3 u.$$

(7) $u = x^x$: $\dfrac{d^3 u}{dx^3} = \dfrac{x^x}{x^2} \{x^2 (\log ex)^3 + 3x(\log ex) - 1\}$.

(8) $\sin(a+bx) = \sin a + bx \cos a - \dfrac{b^2 x^2}{2} . \sin a - \dfrac{b^3 x^3}{2.3} \cos a + \&c.$

(9) $\cos(a+bx) = \cos a - bx \sin a - \dfrac{b^2 x^2}{2} . \cos a + \dfrac{b^3 x^3}{2.3} \sin a + \&c.$

(10) $\log(a + bx) = \log a + \dfrac{bx}{a} - \dfrac{b^2 x^2}{2a^2} + \dfrac{b^3 x^3}{3a^3} - \&c.$

(11) $\sec x = 1 + \dfrac{x^2}{2} + \dfrac{5x^4}{2.3.4} + \&c.$

(12) $(\cos x)^3 = 1 - \dfrac{3x^2}{2} + \dfrac{7x^4}{8} - \&c.$

(13) $(\tan x)^4 = x^4 + \dfrac{4}{3} x^6 + \dfrac{6}{5} x^8 + \&c.$

(14) $e^{\sin x} = 1 + x + \dfrac{x^2}{2} - \dfrac{3x^4}{2.3.4} - \dfrac{8x^5}{2.3.4.5} - \&c.$

(15) $e^x \sec x = 1 + x + x^2 + \dfrac{2x^3}{3} + \&c.$

(16) $\sqrt{e^x + 1} = \sqrt{2}\left(1 + \dfrac{1}{2^2} . \dfrac{x}{1} + \dfrac{3}{2^4} . \dfrac{x^2}{1.2} + \dfrac{7}{2^6} \dfrac{x^3}{2.3} + \&c.\right)$

(17) $\mathrm{Sin}\,(a + bx + cx^2) = \sin a + bx \cos a$
$+ \dfrac{x^2}{2}(2c \cos a - b^2 \sin a) - \dfrac{x^3}{2.3}(6bc \sin a + b^3 \cos a) - \&c.$

(18) If $\cos(m) - \cos(m + y) = x$,

$$y = \frac{x}{\sin m} - \tfrac{1}{2} \cot m . \left(\frac{x}{\sin m}\right)^2 + \&c.$$

CHAPTER IV.

Taylor's Theorem.

60. If $u = f(x)$, and $u_1 = f(x+h)$, then by Taylor's Theorem, so called from its inventor,

$$u_1 = u + \frac{du}{dx} h + \frac{d^2u}{dx^2} \frac{h^2}{1.2} + \frac{d^3u}{dx^3} \frac{h^3}{2.3} + \&c. + \frac{d^nu}{dx^n} . \frac{h^n}{2.3 \ldots n} + \&c.$$

The proof of this theorem may be made to depend upon the proposition, that if $u_1 = f(x+h)$; $\frac{du_1}{dx} = \frac{du_1}{dh}$,

or the coefficient of h is the same in the expansion $f(x+2h)$, whether in $f(x+h)$, x become $x+h$, or h become $h+h$.

Let $x + h = x_1$; $\therefore$ $u_1 = f(x_1)$ and $\frac{du_1}{dx_1} = \frac{d.f(x_1)}{dx_1}$.

But $\frac{du_1}{dx_1} = \frac{du_1}{dx} . \frac{dx}{dx_1} =$ also $\frac{du_1}{dh} . \frac{dh}{dx_1}$.

But h constant, $\frac{dx}{dx_1} = 1$; x constant, $\frac{dh}{dx_1} = 1$;

$$\therefore \frac{du_1}{dx} = \frac{du_1}{dh}.$$

$$\text{Hence } \frac{d.\left(\frac{du_1}{dx}\right)}{dx} = \frac{d.\left(\frac{du_1}{dh}\right)}{dh}; \text{ or } \frac{d^2u_1}{dx^2} = \frac{d^2u_1}{dh^2};$$

$$\text{and } \frac{d^nu_1}{dx^n} = \frac{d^nu_1}{dh^n}.$$

61. Let $\therefore$ $u_1 = u + \frac{du}{dx} h + Ph^2 + Qh^3 + Rh^4 + \&c.$

Since we have proved in Art. 8, that the indices of h in the expansion of $f(x+h)$ are the natural numbers,

$$\therefore \frac{du_1}{dx} = \frac{du}{dx} + \frac{d^2u}{dx^2} h + \frac{dP}{dx} h^2 + \frac{dQ}{dx} h^3 + \&c.$$

$$\frac{du_1}{dh} = \frac{du}{dx} + 2Ph + 3Qh^2 + 4Rh^3 + \&c.$$

whence equating the coefficients of the same powers of h,

$$2P = \frac{d^2u}{dx^2}; \quad \therefore P = \tfrac{1}{2} . \frac{d^2u}{dx^2},$$

$$3Q = \frac{dP}{dx} = \frac{1}{2} \cdot \frac{d^3u}{dx^3}; \quad \therefore \; Q = \frac{1}{2 \cdot 3} \cdot \frac{d^3u}{dx^3},$$

$$4R = \frac{dQ}{dx} = \frac{1}{2 \cdot 3} \cdot \frac{d^4u}{dx^4}; \quad \therefore \; R = \frac{1}{2 \cdot 3 \cdot 4} \cdot \frac{d^4u}{dx^4};$$

$$\therefore \; u_1 = u + \frac{du}{dx} h + \frac{d^2u}{dx^2} \frac{h^2}{1 \cdot 2} + \frac{d^3u}{dx^3} \frac{h^3}{2 \cdot 3} + \&c.$$

COR. From the theorem of Taylor we may deduce that of Maclaurin.

For making $x = 0$, u_1 becomes $f(h)$ and u, $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$, $\frac{d^3u}{dx^3}$, &c. become U_0, U_1, U_2, U_3, &c.;

$$\therefore f(h) = U_0 + U_1 h + U_2 \frac{h^2}{1 \cdot 2} + U_3 \frac{h^3}{2 \cdot 3} + \&c.$$

or putting x for h, in which case u may be put for $f(x)$,

$$u = U_0 + U_1 x + U_2 \frac{x^2}{1 \cdot 2} + U_3 \frac{x^3}{2 \cdot 3} + \&c.$$

Examples.

62. To expand $\sin(x+h)$, $\cos(x+h)$, $\log(x+h)$ and $(x+h)^n$, by Taylor's Theorem, .

$$u_1 = u + \frac{du}{dx} h + \frac{d^2u}{dx^2} \frac{h^2}{1 \cdot 2} + \frac{d^3u}{dx^3} \frac{h^3}{2 \cdot 3} + \&c.$$

(1) $u = \sin x$;

$$\therefore \; \frac{du}{dx} = \cos x, \quad \frac{d^2u}{dx^2} = -\sin x, \quad \frac{d^3u}{dx^3} = -\cos x, \quad \frac{d^4u}{dx^4} = \sin x,$$

after which the values recur;

$$\therefore \; u_1 = \sin(x+h) = \sin x + \cos x \cdot h - \sin x \frac{h^2}{1 \cdot 2} - \cos x \frac{h^3}{2 \cdot 3}$$
$$+ \sin x \frac{h^4}{2 \cdot 3 \cdot 4} + \cos x \frac{h^5}{2 \cdot 3 \cdot 4 \cdot 5} - \&c.$$

(2) $u = \cos x$;

$$\frac{du}{dx} = -\sin x, \quad \frac{d^2u}{dx^2} = -\cos x, \quad \frac{d^3u}{dx^3} = \sin x, \quad \frac{d^4u}{dx^4} = \cos x,$$

$$\therefore \; u_1 = \cos(x+h) = \cos x - \sin x \cdot \frac{h}{1} - \cos x \frac{h^2}{1 \cdot 2}$$
$$- \sin x \cdot \frac{h^3}{2 \cdot 3} + \cos x \cdot \frac{h^4}{2 \cdot 3 \cdot 4} - \&c.$$

COR. If in the two expansions we make $x=0$, we have

$$\sin h = h - \frac{h^3}{2.3} + \frac{h^5}{2.3.4.5} - \&c.$$

$$\cos h = 1 - \frac{h^2}{1.2} + \frac{h^4}{2.3.4} - \&c.$$

(3) $u = \log(x)$;

$$\therefore \frac{du}{dx} = \frac{1}{x} = x^{-1}, \quad \frac{d^2u}{dx^2} = -x^{-2}, \quad \frac{d^3u}{dx^3} = 2x^{-3}, \quad \frac{d^4u}{dx^4} = -2.3x^{-4};$$

$$\therefore u_1 = \log(x+h) = \log x + \frac{h}{x} - \tfrac{1}{2}.\frac{h^2}{x^2} + \tfrac{1}{3}.\frac{h^3}{x^3} - \tfrac{1}{4}.\frac{h^4}{x^4} + \&c.$$

Let $x=1$; $\therefore \log x = 0$;

$$\therefore \log(1+h) = h - \tfrac{1}{2}h^2 + \tfrac{1}{3}h^3 - \tfrac{1}{4}h^4 + \tfrac{1}{5}h^5 - \&c.$$

(4) $u = x^n$;

$$\therefore \frac{du}{dx} = nx^{n-1}, \quad \frac{d^2u}{dx^2} = n(n-1)x^{n-2}, \quad \frac{d^3u}{dx^3} = n.(n-1)(n-2).x^{n-3};$$

$$\therefore u_1 = (x+h)^n = x^n + n.x^{n-1}h + \frac{n.(n-1)}{1.2}.x^{n-2}h^2$$
$$+ \frac{n(n-1)(n-2)}{1.2.3}.x^{n-3}h^3 + \&c.$$

63. The following Proposition is used in some demonstrations of the parallelogram of forces. Given that

$$f(x).f(h) = f(x+h) + f(x-h); \text{ find } f(x).$$

Let u be put for $f(x)$;

$$\therefore u.f(h) = 2\{u + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \frac{d^4u}{dx^4}\frac{h^4}{2.3.4} + \&c.\};$$

$$\therefore f(h) = 2\{1 + \frac{1}{u}\frac{d^2u}{dx^2}\frac{h^2}{1.2} + \frac{1}{u}\frac{d^4u}{dx^4}\frac{h^4}{2.3.4} + \&c.\}.$$

Now since h is independent of x, the coefficients $\frac{1}{u}.\frac{d^2u}{dx^2}$, $\frac{1}{u}\frac{d^4u}{dx^4}$, &c., which cannot contain h, must be constant.

Let $\frac{1}{u}\frac{d^2u}{dx^2} = -a^2$; $\therefore \frac{d^2u}{dx^2} = -a^2u$; $\frac{d^4u}{dx^4} = -a^2\frac{d^2u}{dx^2} = a^4u$.

Hence $f(h) = 2\{1 - \frac{a^2h^2}{1.2} + \frac{a^4h^4}{2.3.4} - \&c.\} = 2\cos ah$,

and $\therefore f(x) = 2\cos ax$; and $f(x \pm h) = 2\cos(ax \pm ah)$,

which may be verified by the formula

$$2\cos A\,.\,2\cos B = 2\cos(A+B) + 2\cos(A-B).$$

64. Taylor's Theorem may be used to approximate to the roots of equations.

Let $X=0$ be an equation, of which x is one of the roots, and a an approximate value of x, so that $x=a+h$, h being a very small quantity, hence since $X=0$ is a function of x;

$$\therefore\ X = 0 = f(a+h) = f(a) + \frac{d\,.f(a)}{da}h + \frac{d^2\,.f(a)}{da^2}\,\frac{h^2}{1\,.\,2} + \&c.;$$

but since h is assumed very small, we may neglect the terms after the second, and so obtain an approximate value of h;

$$\therefore\ 0 = f(a) + \frac{d\,.f(a)}{da}h;\ \therefore\ h = \frac{-f(a)}{\frac{d\,.f(a)}{da}} = \frac{-f(a)}{p},$$

$$\text{and } x = a - \frac{f(a)}{p}.$$

If this value of x be not sufficiently near the true one, let it be put $=a_1$, and the process repeated: we shall at length arrive at results more and more near the true one.

Ex. 1. $x^3-3x+1=0$. By trial 1.5 is found to be near one of the roots.

$$f(a) = a^3 - 3a + 1 = (1.5)^3 - 3\times(1.5) + 1 = -.125,$$

$$\frac{d\,.f(a)}{da} = 3a^2 - 3 = 6.75 - 3 = 3.75;$$

$$\therefore\ h = \frac{.125}{3.75} = .033;\ \therefore\ x = 1.5 + .033 = 1.533.$$

Ex. 2. $x^x = 100$. Since $3^3 = 27$ and $4^4 = 256$; x lies between 3 and 4; let $a = 3.5$.

$$\text{Now } x\log x - \log 100 = 0 = u;\ \therefore\ 1 + \log x = \frac{du}{dx};$$

$$\therefore f(a) = 3.5\log(3.5) - \log 100;\ \frac{d\,.f(a)}{da} = 1 + \log(3.5).$$

$$\text{But } \log_e 100 = 4.60517;\ \log_e 3.5 = 1.25276;$$

$$\therefore f(a) = 3.5\times 1.25276 - 4.60517 = -.22051;\ \frac{d\,.f(a)}{da} = 2.25276;$$

$$\therefore\ h = \frac{.22051}{2.25276} = .09832;\ \text{and } x = a + h = 3.59832;$$

a more exact value may be obtained by putting 3.59832 for a.

The Napierian logarithms are obtained from the common logarithms by dividing each logarithm by the number .43429.

$$\text{Thus } \log_e 100 = \frac{2}{.43429} = 4.60517.$$

65. Transform the equation $x^n - px^{n-1} + qx^{n-2} - \&c. = 0$, into one whose roots shall be diminished by a constant quantity z.

Let $x = z + y$; $X = f(z + y)$, and let $Z = f(z)$;

$$\therefore X = Z + \frac{dZ}{dz}y + \frac{d^2Z}{dz^2}\cdot\frac{y^2}{1.2} + \frac{d^3Z}{dz^3}\cdot\frac{y^3}{2.3} + \&c. = 0.$$

Or if Z_1, Z_2, Z_3, &c....Z_n, be put for the differential coefficients, the transformed equation becomes

$$Z + Z_1y + \frac{Z_2y^2}{1.2} + \frac{Z_3y^3}{2.3} + \&c. + \frac{Z_{n-1}y^{n-1}}{1.2...(n-1)} + \frac{Z_ny^n}{2.3...n} = 0,$$

where Z is the value of X, when z is put for x;

$$\therefore Z = z^n - pz^{n-1} + qz^{n-2} - \&c.$$

$$\text{and } Z_1 = nz^{n-1} - (n-1)pz^{n-2} + (n-2)qz^{n-3} - \&c.$$

$$\vdots$$

$$Z_{n-1} = n(n-1)(n-2)...3.2z - (n-1).(n-2)...2.p,$$

$$\text{and } Z_n = n(n-1)(n-2)...3.2;$$

therefore, after writing the terms in an inverse order,

$$y^n + (nz - p)y^{n-1} + \&c. + Z = 0.$$

Cor. This equation is used to take away any particular term of an equation, by putting any of the coefficients Z_1, Z_2, &c. $= 0$, and substituting in the others the value of z derived from it.

Ex. Transform $3x^3 + 15x^2 + 25x - 3 = 0$ into an equation wanting the second term.

$$\text{Here } Z + Z_1y + \frac{Z_2y^2}{1.2} + \frac{Z_3y^3}{2.3} = 0, \text{ (for } Z_4 = 0),$$

$$Z = 3z^3 + 15z^2 + 25z - 3,$$

$$Z_1 = 9z^2 + 30z + 25,$$

$$Z_2 = 18z + 30,$$

$$Z_3 = 18, \text{ and } Z_4 = 0.$$

But $Z_2=0$; $\therefore\ z=-\frac{30}{18}=\frac{-5}{3}$;

$$Z_1=25-50+25=0,$$

$$Z=-\frac{125}{9}+\frac{125}{3}-\frac{225}{3}-3=\frac{-152}{9};$$

$$\therefore\ -\frac{152}{9}+\frac{18y^2}{6}=0;\ \therefore\ y^2-\frac{152}{27}=0.$$

Examples.

(1) $\operatorname{Tan}(x+h)=\tan x+h\,.\sec^2 x+h^2\,.\tan x\,.\sec^2 x+\&c.$

(2) $$\operatorname{Sin}^{-1}(x+h)=\sin^{-1}x+\frac{h}{\sqrt{1-x^2}}+\frac{h^2x}{2(1-x^2)^{\frac{3}{2}}}+\frac{(1+2x^2)h^3}{2\,.\,3(1-x^2)^{\frac{5}{2}}}+\&c.$$

(3) $$\operatorname{Tan}^{-1}(x+h)=\tan^{-1}x+\frac{1}{1+x^2}h-\frac{2x}{(1+x^2)^2}\cdot\frac{h^2}{2}+\frac{2(3x^2-1)}{(1+x^2)^3}\,\frac{h^3}{2\,.\,3}-\&c.$$

(4) Prove that if $u=f(x)$,

$$f\left(\frac{x}{2}\right)=u-\frac{du}{dx}\,\frac{x}{2}+\frac{d^2u}{dx^2}\cdot\frac{x^2}{2\,.\,2^2}-\frac{d^3u}{dx^3}\cdot\frac{x^3}{2\,.\,3\,.\,2^3}+\&c.$$

$$f\left(\frac{x}{1+x}\right)=u-\frac{du}{dx}\cdot\frac{x^2}{1+x}+\frac{d^2u}{dx^2}\cdot\frac{x^4}{2(1+x)^2}-\frac{d^3u}{dx^3}\cdot\frac{x^6}{2\,.\,3(1+x)^3}+\&c$$

(5) Approximate to a root of the equations

(1) $x^3-12x-28=0$. Ans. $x=4.302$.

(2) $x^4+x-3=0$. Ans. $x=1.165$.

(6) If $u=f(x)$, and if when $x=a$; $u=b$; then

$$u=b+\frac{du}{dx}(x-a)-\frac{d^2u}{dx^2}\cdot\frac{(x-a)^2}{2}+\frac{d^3u}{dx^3}\cdot\frac{(x-a)^3}{2\,.\,3}-\&c.$$

In $f(x+h)$, for $x+h$, put a and the theorem (Euler's) is found.

CHAPTER V.

Failure of Taylor's Theorem: Limits of the same Theorem.

66. By Taylor's Theorem we can expand $f(x+h)$ into the series

$$f(x) + ph + qh^2 + rh^3 + \&c.$$

where the powers of h are integral and ascend.

Indeed we have proved, (Art. 8), that so long as x retains its general value, the expansion of $f(x+h)$ cannot contain fractional powers of h. As this proposition is an important one, we here give the reasoning commonly made use of to establish its truth.

Assume $f(x+h) = u + Ph^{\frac{m}{n}} + R$,

where R represents the sum of all the other terms.

Then since $x+h$ enters $f(x+h)$ in the same manner as x enters $f(x)$, it is plain that both functions (undeveloped) have the same number of values, and that the developement of $f(x+h)$ ought to contain no more than $f(x)$ or $f(x+h)$ does. But if particular values be given to x, which will make P neither infinite nor evanescent; then to each value of x there will correspond n values of $Ph^{\frac{m}{n}}$, since $h^{\frac{m}{n}}$ has n different values; and consequently the expanded function will contain n times as many values as the unexpanded one; and therefore n times as many as $f(x)$, which is manifestly contradictory.

67. If then such a value as a given to x in $f(x+h)$ makes the unexpanded function $f(a+h)$ contain fractional powers of h, we cannot expect that Taylor's Theorem will give the required developement. Now the hypothesis that $x=a$ introduces a fractional index of h into $f(x+h)$, supposes that in the original function there must have been some such term as $(x-a)^{\frac{m}{n}}$, which becomes $(x-a+h)^{\frac{m}{n}}$ in u_1, or $h^{\frac{m}{n}}$ when $x=a$. In such a case it is clear that some of the differential coefficients will become infinite, when $x=a$.

$$\text{Thus if, } u = b + (x-a)^{\frac{m}{n}};$$

$$\therefore \frac{du}{dx} = \frac{m}{n}(x-a)^{\frac{m}{n}-1};$$

$$\frac{d^2u}{dx^2} = \frac{m}{n}\left(\frac{m}{n}-1\right)(x-a)^{\frac{m}{n}-2};$$

$$\therefore u_1 = b + (x-a)^{\frac{m}{n}} + \frac{m}{n}\cdot(x-a)^{\frac{m}{n}-1} h$$

$$+ \frac{m}{n}\cdot\left(\frac{m}{n}-1\right)\cdot(x-a)^{\frac{m}{n}-2}\frac{h^2}{1\cdot 2} + \&c.$$

$$+ \frac{m}{n}\cdot\left(\frac{m}{n}-1\right)\dots\left(\frac{m}{n}-p+1\right)(x-a)^{\frac{m}{n}-p}\frac{h^p}{1\cdot 2\dots p} + \&c.$$

where if $\frac{m}{n} < p$ and $> p-1$, the $(p+1)^{\text{th}}$ term and all that follow it will become infinite when $x = a$.

This circumstance of the differential coefficients becoming infinite when $x = a$ is called the *Failure* of Taylor's Theorem, an improper phrase, since it is rather an index that the function cannot be expanded in integral powers of h.

68. Again, as the general expansion of $f(x+h)$ never contains negative powers of h, for if $f(x+h)$ could

$$= A + Bh^{-a} + \&c.$$

if $h = 0$, $f(x+h)$ instead of becoming $f(x)$, would be infinite, we may be led to expect that if $x = a$ introduces into the unexpanded function $f(x+h)$ a term involving h^{-n}, the expansion by Taylor's Theorem will indicate some absurdity. Now it is clear that to have such a term dependent on h^{-n}, we must originally have had such a term as $\frac{M}{(x-a)^n}$; for putting $x+h$ for x, $\frac{M}{\overline{x-a}|^n}$ becomes $\frac{M}{(x+h-a)^n} = \frac{M}{h^n}$, when $x = a$. M not being supposed to vanish when $x = a$.

Here all the derivatives of $\frac{1}{(x-a)^n}$ are infinite when $x = a$.

69. The theorem therefore fails whenever $x = a$ makes some surd disappear from $u = f(x)$, and therefore introduces into $u_1 = f(x+h)$, a term involving a fractional power of h; or when $x = a$ renders the original function infinite.

As an example of the first case, let $u = b + \sqrt{x-a}$;

$$\therefore\ u_1 = b + \sqrt{x+h-a}$$

$$= b + \sqrt{x-a} + \tfrac{1}{2}\cdot\frac{1}{\sqrt{x-a}}\,h - \tfrac{1}{4}\cdot\frac{1}{(x-a)^{\frac{3}{2}}}\,\frac{h^2}{1\cdot 2} + \&c.$$

Make $x = a$; $\therefore\ u = b$, $u_1 = b + \sqrt{h}$,

and the expanded function contains infinite terms.

As an example of the second case, let $u = \dfrac{1}{x-a}$;

$$\therefore\ u_1 = \frac{1}{x-a+h} = \frac{1}{x-a} - \frac{h}{(x-a)^2} + \frac{h^2}{(x-a)^3} - \&c.$$

where $u = \infty$, $u_1 = \dfrac{1}{h}$, and the terms of the expanded function are infinite, when $x = a$.

70. Should however $f(a+h)$ contain, when expanded, integral powers of h as far as the $(n-1)^{\text{th}}$, and afterwards fractional powers, the first (n) coefficients may be found by means of Taylor's Theorem.

Let $f(a+h) = A + Bh + Ch^2 + \&c. + Nh^{n-1} + Ph^{a} + \&c.$ where a is a fraction between $n-1$ and n.

Now since the coefficients A, B, C, N, do not contain h we may obtain their values by Maclaurin's Theorem, by finding the differential coefficients of $f(a+h)$ with respect to h, and then making $h = 0$; thus

$$\frac{d\,.f(a+h)}{dh} = B + 2Ch + \&c. + (n-1)Nh^{n-2} + aPh^{a-1} + \&c.$$

$$\frac{d^2.f(a+h)}{dh^2} = 2C + \&c. + (n-1)(n-2)Nh^{n-3} + a(a-1)Ph^{a-2} + \&c.$$

$$\vdots$$

$$\frac{d^{n-1}.f(a+h)}{dh^{n-1}} = \ldots(n-1)(n-2)\ldots 2.1N + a.(a-1)\ldots(a-n+2)Ph^{a-n+1},$$

$$\frac{d^n.f(a+h)}{dh^n} = a(a-1)(a-2)\ldots(a-n+1).Ph^{a-n} + \&c.$$

Now if $h = 0$, since $a > n-1$, but $< n$, the terms involving P will vanish from the first $(n-1)$ equations, and the $(n-1)$ differential coefficients will be found.

But since $a - n$ is negative, then when $h = 0$

$$\frac{d^n.f(a+h)}{dh^n} = \frac{a(a-1)(a-2)\ldots(a-n+1).P}{h^{n-a}} \text{ is infinite.}$$

71. Again, should the substitution of $x = a$ introduce negative powers of h, all the differential coefficients will be

infinite. This, as it has been observed, is the case, when $u=f(x)$ contains a term $\frac{A}{(x-a)^m}$, for then, if x become $x+h$,

$$\frac{A}{(x-a)^m}=\frac{A}{(x+h-a)^m}=\frac{A}{h^m} \text{ when } x=a.$$

Let then $f(a+h)=Ah^{-m}+\&c.$;

$$\therefore \frac{d.f(a+h)}{dh}=\frac{-mA}{h^{m+1}}+\&c.,$$

$$\vdots \qquad \vdots$$

$$\text{and } \frac{d^n.f(a+h)}{dh^n}=\frac{-m(m+1)(m+2)\dots(m+n-1).A}{h^{m+n}},$$

which it is manifest becomes infinite if $h=0$.

72. Hence if the n^{th} derivative become infinite when $x=a$, the true expansion contains a fractional power of h lying between $(n-1)$ and (n); and if $x=a$ makes $f(x)=\infty$, the true expansion contains negative powers of h.

Ex. If $u=bx^m+c(x-a)^{\frac{p}{q}}$.

$$\frac{du}{dx}=m.bx^{m-1}+\frac{cp}{q}.(x-a)^{\frac{p}{q}-1},$$

$$\vdots$$

$$\frac{d^nu}{dx^n}=m(m-1)\dots(m-n+1)bx^{m-n},$$

$$+c\frac{p}{q}.\left(\frac{p}{q}-1\right)\dots\left(\frac{p}{q}-n+1\right).(x-a)^{\frac{p}{q}-n},$$

and let $\frac{p}{q}<n$ but $>n-1$. Then $\frac{d^nu}{dx^n}$ is the first differential coefficient which becomes infinite, and there ought in the true expansion to be a term involving $h^{\frac{p}{q}}$, which there is; for by putting $x+h$ for x, and afterwards writing a for x, we have $f(a+h)=b.(a+h)^m+ch^{\frac{p}{q}}$.

If $m<n$, the values of the differential coefficients until we come to the n^{th}, will disappear when $x=a$.

73. In functions of this kind recourse must be had to the common algebraical methods.

Thus, suppose $u=2ax+a\sqrt{x^2-a^2}$;

$$\therefore f(a+h)=2a(a+h)+a\sqrt{2ah+h^2}$$

$$=2a(a+h)+a\sqrt{2ah}.\left(1+\frac{h}{2a}\right)^{\frac{1}{2}},$$

$\left(1+\frac{h}{2a}\right)^{\frac{1}{2}}$ is to be expanded by the Binomial Theorem.

74. The Limits of Taylor's Theorem.

If $f(x+h)$ be expanded by Taylor's Theorem, and we stop at the n^{th} term, the sum of the first n terms may differ widely from the true value of $f(x+h)$; it is therefore necessary to calculate the amount or limit of the error which arises from neglecting the remaining terms of the series.

To do this we must premise the following articles.

75. If $u=f(x)=0$ when $x=0$, then u and $\frac{du}{dx}$ will have the same sign while x increases from 0 to a, if a be positive; but contrary signs, if a be negative; $\frac{du}{dx}$ being supposed neither to change its sign, nor to become infinite, while x increases from 0 to a.

Let a be divided into n equal parts, each $=h$, or $a=nh$.

Then since $f(x+h)=f(x)+\frac{du}{dx}h+Ph^2$ (1);

if U_1 and P_1 be the values of $\frac{du}{dx}$ and P, when $x=0$,

$$f(h)=U_1h+P_1h^2.$$

Now if $\left.\begin{matrix}U_2, & U_3, & U_4 \dots U_n \\ P_2, & P_3, & P_4 \dots P_n\end{matrix}\right\}$ be the values of $\frac{du}{dx}$ and P,

when h, $2h$, $3h \dots (n-1)h$ are put for x; $\therefore$ from (1),

$$\begin{aligned} f(h+h)-f(h) &= U_2h+P_2h^2, \\ f(2h+h)-f(h+h) &= U_3h+P_3h^2, \\ &\vdots \\ f\{(n-1)h+h\}-f\{(n-2)h+h\} &= U_nh+P_nh^2; \end{aligned}$$

whence, by addition,

$$f(nh) \text{ or } f(a)=(U_1+U_2+U_3+\&\text{c.}+U_n)h+(P_1+P_2+\&\text{c.}+P_n)h^2;$$

and by diminishing h, the first term $(U_1+U_2+U_3+\&\text{c.}+U_n)h$ may be rendered greater than the second, and therefore the algebraical sign of $f(a)$ will depend only on the first term.

Also $f(h)$ will have the same sign as U_1, which is $\frac{du}{dx}$ when $x=0$; or, since $\frac{du}{dx}$ does not change its sign, $f(h)$

will have the same sign as $\frac{du}{dx}$. Also $f(2h)-f(h)$ will have the same sign as U_2, which is the value of $\frac{du}{dx}$ when $x=h=\frac{a}{n}$; and therefore the same sign as $\frac{du}{dx}$.

And therefore $f(a)$ which has the same sign as the sum of the products $(U_1+U_2+U_3+\&c.+U_n)\frac{a}{n}$ will have the same sign as $\frac{du}{dx}$, if a be positive, but the contrary sign if a be negative.

76. Let $f(x)$ and $\phi(x)$ be two functions of x, x_1 and x_1+h two given values of x, to find the value of the ratio

$$\frac{f(x_1+h)-f(x_1)}{\phi(x_1+h)-\phi(x_1)};$$

it being supposed that the functions $f(x)$ and $\phi(x)$, constantly increase or constantly decrease, for every value of x from $x=x_1$ to $x=x_1+h$, or in other words that their derivatives have constantly the same algebraic sign.

Suppose therefore that the derivative of $f(x)$ or $f'(x)$ is always positive between those limits: and let A and B be the greatest and least value of the ratio $\frac{f'(x)}{\phi'(x)}$ between x_1 and x_1+h; hence

$$\frac{f'(x)}{\phi'(x)}<A \text{ and } >B;$$

$$\therefore f'(x)-A\,.\,\phi'(x)<0, \text{ and } f'(x)-B\,.\,\phi'(x)>0.$$

But $f'(x)-A\phi'(x)$ is the derivative of $f(x)-A\phi(x)$: and therefore this function constantly decreases from x_1 to x_1+h:

$$\therefore f(x_1+h)-A\phi(x_1+h)<f(x_1)-A\phi(x_1);$$

$$\therefore \frac{f(x_1+h)-f(x_1)}{\phi(x_1+h)-\phi(x_1)}<A.$$

Similarly: $$\frac{f(x_1+h)-f(x_1)}{\phi(x_1+h)-\phi(x_1)}>B.$$

Then since the ratio of $\frac{f'(x)}{\phi'(x)}$ is continuous between the given values of x, there must be some value of x, lying between x_1 and x_1+h, which will make it equal to $\frac{f(x_1+h)-f(x_1)}{\phi(x_1+h)-\phi(x_1)}$

which lies between the greatest and least values of $\dfrac{f'(x)}{\phi'(x)}$. Let this value of x be $x_1 + \theta h$ where θ is < 1. Then

$$\frac{f(x_1+h)-f(x_1)}{\phi(x_1+h)-\phi(x_1)}=\frac{f'(x_1+\theta h)}{\phi'(x_1+\theta h)}.$$

If $f'(x)$ had been constantly negative, the inequalities will exchange their values, but the result will be the same.

77. If there be some value of x as x_1, which makes $f(x_1)=0$, and also $\phi(x_1)=0$, the formula becomes

$$\frac{f(x_1+h)}{\phi(x_1+h)}=\frac{f'(x_1+\theta h)}{\phi'(x_1+\theta h)}=\frac{f'(x+h_1)}{\phi'(x+h_1)},\quad h_1=\theta h.$$

If also $f'(x_1)=0$ and $\phi'(x_1)=0$, then similarly,

$$\frac{f'(x+h_1)}{\phi'(x+h_1)}=\frac{f^2.(x_1+h_2)}{\phi^2.(x_1+h_2)},\quad h_2 \text{ being} < h_1;$$

$$\therefore \frac{f(x_1+h)}{\phi(x_1+h)}=\frac{f^2.(x_1+h_2)}{\phi^2.(x_1+h_2)}:$$

and then finally, if also $f^2(x)=0$, $\phi^2(x_1)=0$; ......

and $f^{n-1}(x_1)=0$, and $\phi^{n-1}(x_1)=0$,

$$\frac{f(x_1+h)}{\phi(x_1+h)}=\frac{f^n.(x_1+\theta h)}{\phi^n.(x_1+\theta h)}.$$

78. If as an example we make $\phi(x)=(x-x_1)^n$;

and $\therefore$ $\phi(x_1+h)=h^n$; but $\phi'(x_1)=0\ldots\phi^{n-1}(x_1)=0$;

$\phi^n.(x_1+h)=n(n-1)(n-2)\ldots3.2.1.=\therefore\ \phi^n.(x_1+\theta h)$.

And if at the same time $f'(x_1)=0\ldots f^{n-1}(x_1)=0$,

$$\text{then } f(x_1+h)-f(x_1)=\frac{h^n}{1.2.3\ldots n},\ f^n.(x_1+\theta h).$$

Hence, if h be a small quantity tending to zero, or as it is called an infinitesimal, and if at the same time $f^n.(x_1)$ is finite: the increment of $f(x_1)$ will be an infinitesimal of the n^{th} order.

If also $f(x_1)=0$ the formula becomes

$$f(x_1+h)=\frac{h^n}{2.3.n}.f^n.(x_1+\theta h);$$

and if now we have at the same time $x_1=0$,

$$f(h)=\frac{h^n}{2.3\ldots n}.f^n.(\theta h),\text{ or writing } x \text{ for } h,$$

$$f(x)=\frac{x^n}{2.3.n}.f^n.(\theta x);$$

in such a case as this, in which x is an infinitesimal, $f(x)$ will be one of the n^{th} order, if $f^n(\theta x)$ be finite.

Ex. Let $u = x - \sin x$, x tending to zero.

Here $x = 0$, $f(x) = 0$, $f'(x) = 1 - \cos x = 0$; $f''(x) = \sin x = 0$,

$$f'''(x) = \cos x = 1\,; \quad \therefore\ u = \frac{x^3}{2\,.\,3}\,.f'''(\theta x) = \frac{x^3}{2\,.\,3},$$

and is an infinitesimal of the third order.

79. We now proceed to determine the value of R in Taylor's Theorem, R being the remainder after n terms.

For R put $\phi(h)$.

$$\therefore f(x+h) = f(x) + h\,.f'(x) + \frac{h^2}{2}f''(x) + \ldots + \frac{h^{n-1}}{2\ldots(n-1)}f^{n-1}(x) + \phi(h).$$

Now the differential coefficients of both members of this equation with regard to h are identical; and since it is obvious that $f(h)$ (being a term multiplied by h^n) vanishes when $h = 0$, as do also its $(n-1)$ derivatives;

$$\therefore\ \phi^n(h) = f^n\,.\,(x+h)\,;$$

$$\therefore\ \frac{h^n}{2\,.\,3\,.\,n}\,\phi^n\,.\,(\theta h) = \frac{h^n}{2\,.\,3\,.\,n}\,.f^n(x+\theta h).$$

$$\text{But } \phi(h) = \frac{h^n}{2\,.\,3\ldots n}\,.\,\phi^n(\theta h) = \therefore\ \frac{h^n}{2\,.\,3\,.\,n}\,.f^n(x+\theta h)\,;$$

$$\therefore\ u_1 = f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \&c. + \frac{h^n}{2.3.n}\,.f^n.(x+\theta h);$$

which is the complete form of Taylor's Theorem.

Cor. If $x = 0$, and if we write x for h, we may deduce the theorem of Maclaurin, and exhibit the value of the remainder: for

$$f(x) = u_0 + u_1 x + u_2\frac{x^2}{2} + u_3\frac{x^3}{2\,.\,3} + \&c. + \frac{x^n}{2\,.\,3\,.\,n}\,.f^n(\theta x).$$

Neither in this nor in Taylor's Theorem is it necessary to know the exact value of θ, only that it must be between 0 and 1.

Ex. Let $u = a^x$; find R after n terms, when $u_1 = a^{x+h}$;

$$f^n(a^x) = A^n\,.\,a^x\,; \quad \therefore\ R = \frac{A^n h^n}{2\,.\,3\ldots n}\,.\,a^{x+\theta h} = \frac{A^n\,.\,h^n}{2\,.\,3\ldots n}\,.\,a^x a^{\theta h};$$

if the same be required in the expansion of a^x,

$$R = \frac{x^n}{2\,.\,3\,.\,n}f^n\,.\,(\theta x) = \frac{A^n\,.\,x^n}{2\,.\,3\ldots n}\,.\,a^{\theta x}.$$

CHAPTER VI.

Vanishing Fractions.

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80. When the substitution of a particular value for the unknown quantity, makes both the numerator and denominator of a fraction vanish, the fraction is called a vanishing fraction.

Thus $\frac{x^2-1}{x-1}$ becomes $\frac{0}{0}$ when $x=1$, but since by division, $\frac{x^2-1}{x-1}=x+1$; the true value of the fraction when $x=1$, is $1+1=2$.

Here both the numerator and denominator vanish if $x=1$, because they both contain the factor $x-1$, which vanishes on the same supposition.

81. That the value of the fraction $\frac{x^2-1}{x-1}$ tends to 2 as its limit as x tends to 1, may be shewn by actual substitution; put for x successively the numbers

$$3,\ 2,\ 1\tfrac{1}{2},\ 1\tfrac{1}{10},\ 1\tfrac{1}{100},\ 1\tfrac{1}{1000},\ \&c.$$

then the values of the fraction become

$$4,\ 3,\ 2\tfrac{1}{2},\ 2\tfrac{1}{10},\ 2\tfrac{1}{100},\ 2\tfrac{1}{1000},\ \&c.$$

which sufficiently shew that as x approaches unity, the value of the fraction approaches 2.

82. We proceed to shew that the values of these fractions may be finite, nothing, or infinite; and thus the term *vanishing fraction*, is used with great impropriety.

(1°) Let $u=\frac{P}{Q}$ be a fraction involving integral powers of x, and no surds; and let $x=a$ make $P=0$, and $Q=0$; then P and Q are both divisible by $x-a$, or its powers*;

$$\therefore \text{ let } P=p\,.\,\overline{x-a}|^m, \text{ and } Q=q\,.\,\overline{x-a}|^n;$$

$$\therefore\ u=\frac{p}{q}\,.\,\frac{\overline{x-a}|^m}{\overline{x-a}|^n}.$$

* For divide P by $x-a$, and let m be the quotient and n the remainder, if any; $\therefore P=(x-a)m+n$; make $x=a$; $\therefore P=0$ by hypothesis; $\therefore 0=m(a-a)+n$; $\therefore n=0$; or P is divisible by $x-a$.

(1) Let $m=n$, $\therefore\ u=\dfrac{p}{q}$; and $=\dfrac{p'}{q'}$ when $x=a$,

which is finite, since neither p nor q contain $x-a$.

(2) Let $m>n$; $\therefore\ u=\dfrac{p}{q}\cdot\overline{x-a}\rceil^{m-n}=0$; if $x=a$.

(3) Let $m<n$; $\therefore\ u=\dfrac{p}{q}\cdot\dfrac{1}{\overline{x-a}\rceil^{n-m}}=\dfrac{1}{0}=\infty$, if $x=a$.

83. Hence it appears the true value of such a fraction is found by getting rid of that power of $(x-a)$, which is common both to the numerator and denominator.

When m and n are whole numbers, the value of the fraction may be found by successive differentiation. For

Let $u=\dfrac{P}{Q}$; $\therefore\ uQ=P$; $\therefore\ u\dfrac{dQ}{dx}+Q\dfrac{du}{dx}=\dfrac{dP}{dx}$.

Let $x=a$; $\therefore\ Q=0$, by hypothesis;

$$\therefore\ u\frac{dQ}{dx}=\frac{dP}{dx};\quad \therefore\ u=\frac{\dfrac{dP}{dx}}{\dfrac{dQ}{dx}},$$

or is equal to the ratio of the differential coefficients of the numerator and denominator, a being put for x.

But if $x=a$, also makes $\dfrac{dP}{dx}=0$ and $\dfrac{dQ}{dx}=0$; then by repeating the process,

$$u=\frac{\dfrac{d}{dx}\cdot\left(\dfrac{dP}{dx}\right)}{\dfrac{d}{dx}\cdot\left(\dfrac{dQ}{dx}\right)}=\frac{\dfrac{d^2P}{dx^2}}{\dfrac{d^2Q}{dx^2}},$$

and the differentiation must be continued until one of the differential coefficients becomes finite when $x=a$.

If both be finite at the same time the fraction is finite; it is nothing, if the differential coefficient of the denominator be first finite; and infinite when that is the case with the differential coefficient of the numerator.

84. If fractional powers of $x-a$, be also found in the numerator and denominator, this method is inapplicable, since $x=a$ will make one of the differential coefficients infinite.

Thus, if $u = \dfrac{(x^2-a^2)^{\frac{1}{2}}}{\sqrt{x-a}} = \sqrt{x+a} = \sqrt{2a}$;

$$\frac{dP}{dx} = \frac{x}{\sqrt{x^2-a^2}}, \text{ and } \frac{dQ}{dx} = \frac{1}{2\sqrt{x-a}},$$

both of which become infinite when $x = a$.

In such a case we may use the following method.

(2°) Let $\dfrac{P}{Q}$ be the fraction, where P and Q vanish if $x = a$. For x put $a+h$, and expand the numerator and denominator by the Binomial Theorem according to the increasing powers of h, so that the fraction becomes

$$\frac{Ah^{\alpha} + Bh^{\beta} + Ch^{\gamma} + \&c.}{A_1 h^{\alpha_1} + B_1 h^{\beta_1} + C_1 h^{\gamma_1} + \&c.},$$

which is of the proper form; $\because$ if $h = 0$ it becomes $\dfrac{0}{0}$.

There will be three cases, $\alpha = \alpha_1$, $\alpha > \alpha_1$, and $\alpha < \alpha_1$.

(1) If $\alpha = \alpha_1$ divide each term by h^{α}, and we have

$$\therefore \frac{A + Bh^{\beta-\alpha} + Ch^{\gamma-\alpha} + \&c.}{A_1 + B_1 h^{\beta_1-\alpha} + C_1 h^{\gamma_1-\alpha} + \&c.} = \frac{A}{A_1}, \text{ or finite if } h = 0.$$

(2) $\alpha > \alpha_1$, then the fraction

$$= \frac{Ah^{\alpha-\alpha_1} + Bh^{\beta-\alpha_1} + \&c.}{A_1 + B_1 h^{\beta_1-\alpha_1} + \&c.} = 0, \text{ when } h = 0.$$

(3) $\alpha < \alpha_1$, then

$$\frac{A + Bh^{\beta-\alpha} + \&c.}{A_1 h^{\alpha_1-\alpha} + B_1 h^{\beta_1-\alpha} + \&c.} = \frac{A}{0} = \infty, \text{ when } h = 0.$$

Cor. 1. If $u = \dfrac{P}{Q} = \dfrac{\infty}{\infty}$, when $x = a$, it $= \dfrac{0}{0}$,

$$\text{For } \frac{P}{Q} = \frac{1}{\frac{Q}{P}} = \frac{\frac{1}{Q}}{\frac{1}{P}} = \frac{\frac{1}{\infty}}{\frac{1}{\infty}} = \frac{0}{0}; \text{ when } x = a.$$

Cor. 2. If $u = \dfrac{1}{P} - \dfrac{1}{Q} = \dfrac{1}{0} - \dfrac{1}{0}$; it also $= \dfrac{0}{0}$.

$$\text{For } \frac{1}{P} - \frac{1}{Q} = \frac{Q-P}{PQ} = \frac{0}{0}, \text{ when } x = a.$$

Cor. 3. Next $P \times Q = 0 \times \infty = \dfrac{0}{0}$.

$$\text{For } Q = \frac{1}{Q_1}, \text{ if } Q_1 = 0, \text{ when } x = a;$$

$$\therefore P \times Q = P \times \frac{1}{Q_1} = \frac{P}{Q_1} = \frac{0}{0}, \text{ if } x = a.$$

Ex. 1. Find the value of $u = \dfrac{x^3 - 1}{x^3 + 2x^2 - x - 2}$, when $x = 1$,

$$P = x^3 - 1; \quad \therefore \frac{dP}{dx} = 3x^2 = 3, \text{ when } x = 1,$$

$$Q = x^3 + 2x^2 - x - 2; \quad \therefore \frac{dQ}{dx} = 3x^2 + 4x - 1 = 6, \text{ if } x = 1;$$

$$\therefore u = \frac{0}{0} = \frac{3}{6} = \frac{1}{2}.$$

Ex. 2. Find the value of $\dfrac{a^x - b^x}{x}$, when $x = 0$.

$$P = a^x - b^x, \text{ and } Q = x,$$

$$\frac{dP}{dx} = a^x \log a - b^x \log b = \log a - \log b = \log \frac{a}{b}, \text{ when } x = 0,$$

$$\text{and } \frac{dQ}{dx} = 1; \quad \therefore u = \log\left(\frac{a}{b}\right).$$

Ex. 3. $u = \dfrac{x^x - x}{1 - x + \log x} = \dfrac{0}{0}$, if $x = 1$.

$$P = x^x - x, \text{ and } Q = 1 - x + \log x,$$

$$\frac{dP}{dx} = x^x(1 + \log x) - 1 = 0, \text{ if } x = 1,$$

$$\frac{dQ}{dx} = -1 + \frac{1}{x} = 0, \text{ if } x = 1,$$

$$\frac{d^2P}{dx^2} = x^x(1 + \log x)^2 + \frac{x^x}{x} = 2, \text{ if } x = 1,$$

$$\frac{d^2Q}{dx^2} = -\frac{1}{x^2} = -1, \text{ if } x = 1;$$

$$\therefore u = \frac{2}{-1} = -2.$$

85. We may often dispense with differentiation.

Ex. 4. $u = \dfrac{e^x - 1 - \log(1 + x)}{x^2} = \dfrac{0}{0}$, when $x = 0$.

$$e^x - 1 = x + \frac{x^2}{1.2} + \frac{x^3}{2.3} + \&c.$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.;$$

$$\therefore \frac{e^x - 1 - \log(1+x)}{x^2} = 1 - \frac{x}{6} + \&c. = 1, \text{ if } x = 0.$$

Ex. 5. If $u = \dfrac{1}{1-x} - \dfrac{2}{1-x^2} = \infty - \infty$, when $x = 1$.

$$\frac{1}{1-x} - \frac{2}{1-x^2} = \frac{1+x-2}{1-x^2} = -\frac{(1-x)}{1-x^2} = -\frac{1}{1+x} = -\tfrac{1}{2}, \text{ if } x = 1.$$

Ex. 6. If $u = \dfrac{\pi}{4x} - \dfrac{\pi}{2x(e^{\pi x}+1)}$; find it, if $x = 0$.

$$\text{Here } u = \frac{\pi}{4x} \cdot \frac{e^{\pi x} - 1}{e^{\pi x} + 1} = \frac{0}{0}, \text{ if } x = 0.$$

Expand $e^{\pi x}$ by the formula $e^z = 1 + z + \dfrac{z^2}{1.2} + \&c.$

$$u = \frac{\pi}{4x} \cdot \left(\frac{\pi x + \dfrac{\pi^2 x^2}{1.2} + \&c.}{2 + \pi x + \dfrac{\pi^2 x^2}{1.2} + \&c.} \right) = \frac{\pi}{4} \cdot \left(\frac{\pi + \dfrac{\pi^2 x}{1.2} + \&c.}{2 + \pi x + \dfrac{\pi^2 x^2}{1.2} + \&c.} \right).$$

Let $x = 0$; $\therefore u = \dfrac{0}{0} = \dfrac{\pi^2}{8}$.

Ex. 7. $u = \dfrac{\log x}{x}$; find it, when $x = \infty$.

$$\text{Let } \log x = y; \quad \therefore x = e^y,$$

$$u = \frac{y}{e^y} = \frac{y}{1 + y + \dfrac{y^2}{1.2} + \dfrac{y^3}{2.3} + \&c.}$$

$$= \frac{1}{\dfrac{1}{y} + 1 + \dfrac{y}{2} + \dfrac{y^2}{2.3} + \&c.} = \frac{1}{0+\infty} = \frac{1}{\infty} = 0, \text{ if } y = \infty.$$

Similarly, if $u = \dfrac{\log x}{x^n}$; and $x = \infty$, $u = 0$.

Ex. 8. $u = \dfrac{\sqrt{a^2 - x^2} + a - x}{\sqrt{a-x} + \sqrt{a^3 - x^3}} = \dfrac{0}{0}$, when $x = a$.

$\dfrac{dP}{dx}$ and $\dfrac{dQ}{dx}$ are infinite when $x = a$, and we may use the second method; let $x = a - h$,

$$\therefore u = \frac{\sqrt{2ah - h^2} + h}{\sqrt{h} + \sqrt{h(a^2 + ax + x^2)}} = \frac{\sqrt{2a - h} + \sqrt{h}}{1 + \sqrt{a^2 + ax + x^2}}.$$

Let $x=a$, or $h=0$. Then $u=\dfrac{\sqrt{2a}}{1+\sqrt{3a^2}}$.

We might divide at once by $\sqrt{a-x}$, and then

$$u=\frac{\sqrt{a+x}+\sqrt{a-x}}{1+\sqrt{a^2+ax+x^2}}=\frac{\sqrt{2a}}{1+\sqrt{3a^2}},\ \text{when } x=a.$$

Ex. 9. $u=\dfrac{x^4-x^3+x^2-1}{x^4-2x^3+2x^2-1}=\frac{3}{2}$; $x=1$.

Ex. 10. $u=\dfrac{1-\sin x+\cos x}{\sin x+\cos x-1}=1$; $x=\dfrac{\pi}{2}$.

Ex. 11. $u=\dfrac{ax^2-2acx+ac^2}{bx^2-2bcx+bc^2}=\dfrac{a}{b}$; $x=c$.

Ex. 12. $u=\dfrac{\cos x-\cos mx}{\cos x-\cos nx}=\dfrac{1-m^2}{1-n^2}$; $x=0$.

Ex. 13. $u=\dfrac{x^3-x^2-8x+12}{x^4-9x^2+4x+12}=\dfrac{1}{3}$: $x=2$.

Ex. 14. $u=\dfrac{e^x-e^{\sin x}}{x-\sin x}=1$; $x=0$.

Ex. 15. $u=\dfrac{\tan x-\sin x}{(\sin x)^3}=\dfrac{1}{2}$; $x=0$.

Ex. 16. $u=\dfrac{1-(n+1).x^n+n.x^{n+1}}{(1-x)^2}=\dfrac{n(1+1)}{2}$; $x=1$.

Ex. 17. $u=\dfrac{x^4-5x^3+9x^2-7x+2}{x^4-6x^3+12x^2-10x+3}=\frac{1}{2}$; $x=1$.

Ex. 18. $u=\dfrac{\sin^{-1}x-x}{(\sin x)^3}=\frac{1}{6}$; $x=0$.

Ex. 19. $u=\dfrac{e^x-e^{-x}}{\log(1+x)}=2$; $x=0$.

Ex. 20. $u=\dfrac{a-x-a\log\left(\dfrac{a}{x}\right)}{a-\sqrt{a^2-(a-x)^2}}=-1$; $x=a$.

Ex. 21. $u=(1-x)\tan\dfrac{\pi x}{2}=\dfrac{1-x}{\cot\dfrac{\pi x}{2}}=\dfrac{2}{\pi}$; $x=1$.

Ex. 22. $u=(1-x)\log(1-x)=0$; $x=1$.

Ex. 23. $u = \dfrac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}} = \dfrac{16a}{9}; \quad x = a.$

Ex. 24. $u = \dfrac{1}{2x^2} - \dfrac{\pi}{2x\tan\pi x} = \dfrac{\pi^2}{6}; \quad x = 0.$

Ex. 25. $u = \dfrac{1 + x - x^2 - x^3}{1 + 2x + 2x^2 + 2x^3 + x^4} = 1; \quad x = -1.$

Ex. 26. $u = \dfrac{x^4 + 3x^3 - 7x^2 - 27x - 18}{x^4 - 3x^3 - 7x^2 + 27x - 18}$

$= 10; \text{ if } x = 3; \quad = \dfrac{1}{10}; \text{ if } x = -3.$

Ex. 27. $u = \dfrac{xe^{2x} + 1 - e^{2x} - x}{e^{2x} - 1} = -1; \quad x = 0.$

Ex. 28. $u = \dfrac{x^3 - 4ax^2 + 7a^2x - 2a^3 - 2a^2\sqrt{2ax - a^2}}{x^2 - 2ax - a^2 + 2a\sqrt{2ax - x^2}}$

$= -5a; \quad x = a.$

Ex. 29. $u = \dfrac{x}{x - 1} - \dfrac{1}{\log x} = \tfrac{1}{2}; \quad x = 1.$

Ex. 30. $u = \dfrac{a^{\log x} - x}{\log x} = \log\left(\dfrac{a}{e}\right); \quad x = 1.$

Ex. 31. $u = \dfrac{\pi}{4x} \cdot \tan\dfrac{\pi x}{2} = \dfrac{\pi^2}{8}; \quad x = 0.$

Ex. 32. If $u^4 - 96a^2u^2 + 100a^2x^2 - x^4 = 0$,

if $x = 0$; $\dfrac{du}{dx} = \pm\dfrac{5}{4}\sqrt{\dfrac{2}{3}}.$

Ex. 33. $u = \dfrac{1 - x^2}{2x}\log\left(\dfrac{1 + x}{1 - x}\right) = 1; \quad x = 0.$

Ex. 34. $u = \dfrac{\log\tan x}{\log\tan 2x} = 1; \quad x = 0.$

Ex. 35. $u = \sqrt{a^2 - x^2}\cot\left(\dfrac{\pi}{2}\sqrt{\dfrac{a - x}{a + x}}\right) = \dfrac{4a}{\pi}; \quad x = a.$

Ex. 36. $u = \dfrac{\sqrt{x} - \sqrt{a} + \sqrt{ax - x^2}}{\sqrt{x^3} - \sqrt{a^3} + \sqrt{2a^2 - 3ax + x^2}} = 1; \quad x = a.$

Ex. 37. $u = \dfrac{(x - a)\sqrt{x - c} + \sqrt{x - a}}{\sqrt{2a} - \sqrt{x + a} + \sqrt{x - a}} = 1; \quad x = a.$

Ex. 38. $u = \dfrac{\sqrt{x^2 - a^2} + a - \sqrt{2ax - x^2}}{\sqrt[3]{x^3 - a^3} - \sqrt{x^2 - a^2}} = 0; \quad x = a.$

CHAPTER VII.

Maxima and Minima.

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86. If $u = f(x)$ express the relation between the function u, and the variable x, then if $x = a$ make $f(a)$ greater than *both* $f(a+h)$ and $f(a-h)$; $u = f(a)$ is said to be a maximum; but if $f(a)$ be less than *both* $f(a+h)$ and $f(a-h)$, it is called a minimum.

Hence the value of a function is said to be a *maximum* or *minimum*, according as the particular value is greater or less than the values which immediately precede and follow it.

From this definition it appears, that if a quantity either continually increase or constantly decrease, it does not possess the property of a maximum or minimum. Also as the words maximum or minimum are used in a relative and not in an absolute sense, functions may possess many maxima or minima.

For we may easily conceive that a quantity after having reached a maximum value may decrease to a minimum value, and afterwards again increase, and thus many maxima and minima may exist in the same function, but which it is obvious must succeed in order.

Thus the alternate elevation and depression of the waves of the sea will with regard to a horizontal line give maxima and minima altitudes.

87. In the circle the sine* which $= 0$, when the arc $= 0$, increases as the arc increases, till the arc $= 90^0$, when the sine $=$ radius, from this value it decreases, till at the end of the second quadrant it becomes $= 0$.

At 90^0, therefore, it is a maximum; for any two sines drawn on opposite sides of the sin 90^0, and equidistant from it, will be both less than the radius.

In the parabola, the line drawn from the focus to the vertex, is less than either of two focal distances which can be drawn to the curve on opposite sides of it; it is therefore a minimum.

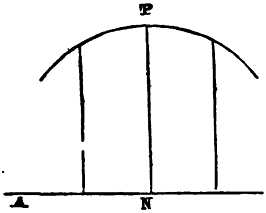
* By the sine is here meant the semichord to which the sine of the angle is proportional.

By reference to figures 1 and 2, we perceive that

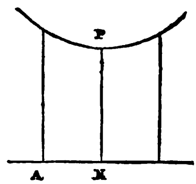
NP in fig. 1, is a maximum,

NP.........2, is a minimum.

(1)



(2)



88. One of the chief applications of the Differential Calculus, is that which affords rules for the discovery of these values.

But the following proposition must first be established.

If $y = A_1 h + A_2 h^2 + A_3 h^3 + \text{\&c.} + A_n h^n + A_{n+1} h^{n+1} + \text{\&c.}$, where the ratio of any coefficient to the one immediately preceding is finite, i.e. $\frac{A_{n+1}}{A_n}$ is finite, h may be so assumed that any one term shall be greater than the sum of all the terms that follow it.

Let r be > the greatest ratio between the cofficients;

$$\therefore \frac{A_2}{A_1} < r, \quad \text{or } A_2 < A_1 r,$$

$$\frac{A_3}{A_2} < r; \quad \therefore A_3 < A_1 r^2,$$

$$\frac{A_4}{A_3} < r; \quad \therefore A_4 < A_1 r^3,$$

&c.

$$\therefore A_1 h + A_2 h^2 + A_3 h^3 + \text{\&c.} < A_1 h + A_1 r h^2 + A_1 r^2 h^3 + \text{\&c.}$$

$$< A_1 h \{1 + rh + r^2 h^2 + \text{\&c.}\}$$

$$< A_1 h \frac{1}{1 - rh}.$$

$$\text{Let } rh = \tfrac{1}{2}, \quad \text{or } h = \frac{1}{2r}; \quad \therefore \frac{1}{1 - rh} = 2;$$

$$\therefore A_1 h + A_2 h^2 + A_3 h^3 + \text{\&c.} < 2A_1 h;$$

$$\therefore A_2 h^2 + A_3 h^3 + \text{\&c.} < A_1 h;$$

similarly may $A_2 h_2$ be shewn to be $> A_3 h^3 + A_4 h^4 + \text{\&c.}$

We have supposed the series to proceed to infinity: if it extend to n terms, it is evident, *à fortiori* that any one term is greater than the sum of all that follow it.

In two of the following series the theorem is, and in one it is not, applicable.

$$x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \&c. \ldots\ldots\ldots (1),$$

$$x + 1.2x^2 + 2.3x^3 + 2.3.4x^4 + \&c. \ldots (2),$$

$$x + 2^2.x^2 + 3^2.x^3 + 4^2.x^4 + \&c. \ldots\ldots\ldots\ldots (3).$$

89. Prop. If $u = f(x)$ be a maximum or minimum when $x = a$. Then on the same supposition, $\frac{du}{dx} = 0$.

Let $u_1 = f(x+h)$, and $u_2 = f(x-h)$.

Now at a maximum or minimum, $u = f(x)$ must be greater or less than both $f(x+h)$, and $f(x-h)$, or greater or less than both u_1 and u_2, and hence, $u_1 - u$ and $u_2 - u$ must both have the same algebraical sign.

$$\text{But } u_1 - u = \frac{du}{dx} h + \frac{d^2u}{dx^2} \frac{h^2}{1.2} + \frac{d^3u}{dx^3} \frac{h^3}{2.3} + \&c.$$

$$\text{and } \therefore u_2 - u = -\frac{du}{dx} h + \frac{d^2u}{dx^2} \frac{h^2}{1.2} - \frac{d^3u}{dx^3} \frac{h^3}{2.3} + \&c.$$

by writing $-h$ for h in the value of $u_1 - u$.

Hence, since the first term of the expansion can be made greater than the sum of all the terms that follow it, (if $x = a$, does not make any of the differential coefficients infinite,) it is clear that whilst the term $\frac{du}{dx} h$ exists, $u_1 - u$ and $u_2 - u$ will have a different algebraical sign: i.e. u_1 and u_2 cannot be both greater or both less than u. Therefore, if there be a maximum or minimum, $\frac{du}{dx} = 0$, and

$$u_1 - u = \frac{d^2u}{dx^2} \frac{h^2}{1.2} + \frac{d^3u}{dx^3} \frac{h^3}{2.3} + \&c.$$

$$u_2 - u = \frac{d^2u}{dx^2} \frac{h^2}{1.2} - \frac{d^3u}{dx^3} \frac{h^3}{2.3} + \&c.$$

Now if $x = a$ does not make $\frac{d^2u}{dx^2} = 0$, the sign of $u_1 - u$ and $u_2 - u$, since h^2 is positive, will depend upon that of $\frac{d^2u}{dx^2}$.

If $\therefore \frac{d^2u}{dx^2}$ be $+$, $u_1 - u$ and $u_2 - u$ are both $+$.

If $\frac{d^2u}{dx^2}$ be $-$, $u_1 - u$ and $u_2 - u$ are both $-$.

If $\frac{d^2u}{dx^2}$ be $+$, u_1 and u_2 are both $> u$; or u is a minimum; if $\frac{d^2u}{dx^2}$ be $-$, u_1 and u_2 are both $< u$; or u is a maximum. Hence this rule. To find whether $u = f(x)$ contain any maxima or minima, put $\frac{du}{dx} = 0$, substitute the values of x thus found in $\frac{d^2u}{dx^2}$, if the results be positive, there are minima; if negative, maxima.

90. Should however $\frac{d^2u}{dx^2} = 0$ when $x = a$,

$$u_1 - u = +\frac{d^3u}{dx^3}\frac{h^3}{2.3} + \&c.$$

$$u_2 - u = -\frac{d^3u}{dx^3}\frac{h^3}{2.3} + \&c.$$

and $u_1 - u$ and $u_2 - u$ have again different signs; and therefore there will be no maxima or minima if $\frac{d^3u}{dx^3}$ exist. Hence it is obvious that there is a maximum or minimum only when the *first differential coefficient that does not vanish is of an even order.*

Cor. 1. If $u =$ maximum or minimum, any constant multiple of u is one also. For if $\frac{du}{dx} = 0$, $a\frac{du}{dx}$ is $= 0$; and $\therefore$ if $u =$ maximum, au is a maximum.

Cor. 2. If $f(x)$ be a maximum or minimum, $\{f(x)\}^n$, where n is integral, is also a maximum or minimum.

For let $u = f(x)$, and $U = \{f(x)\}^n$; but $\frac{du}{dx} = f'(x) = 0$;

$\therefore \frac{dU}{dx} = n\{f(x)\}^{n-1}f'(x) = 0$; or U is a maximum or minimum.

Cor. 3. If $u = f(x)$ be a maximum or minimum, $\log u$ is sometimes a maximum or minimum.

Let $U = \log u$; $\therefore \frac{dU}{dx} = \frac{1}{u}\cdot\frac{du}{dx}$. But $\frac{du}{dx} = 0$; $\therefore \frac{dU}{dx} = 0$, or U is a maximum or minimum, unless $x = a$ makes $u = 0$.

91. If $u=$ maximum, $\frac{1}{u}$ minimum, and conversely.

For let $v=\frac{1}{u}$; $\therefore \frac{dv}{dx}=-\frac{1}{u^2}\frac{du}{dx}$,

$$\frac{d^2v}{dx^2}=\frac{2}{u^3}\cdot\frac{du^2}{dx^2}-\frac{1}{u^2}\cdot\frac{d^2u}{d^2x}=-\frac{1}{u^2}\cdot\frac{d^2u}{dx^2}, \text{ when } u=\text{maximum}.$$

Therefore, if $\frac{d^2u}{dx^2}$ be negative, $\frac{d^2v}{dx^2}$ is positive, or if u be a maximum, $\frac{1}{u}$ is a minimum.

92. In the succeeding examples the following results will be found useful.

(1°). Let $a=$ radius of a circle, then

Area $=\pi a^2$; circumference $=2\pi a$.

Area of sector of a circle $=\frac{1}{2}$ rad $\times$ arc.

Solidity of sphere $=\frac{4}{3}\pi a^3$.

The surface of sphere $=4\pi a^2$.

Convex surface of segment $=2\pi ax$; x being altitude.

(2°). Let $2a$, and $2b$ be the axes of an ellipse;

$\therefore$ area of ellipse $=\pi ab$.

(3°). Let $a=$ axis, $2b$ greatest double ordinate of parabola;

$\therefore$ area $=\frac{2}{3}a\times 2b=\frac{4}{3}ab$.

Solidity of paraboloid $=\frac{1}{2}\pi b^2a$.

(4°). Let $a=$ altitude; $b=$ radius of base of cylinder.

Solidity of cylinder $=\pi b^2a$.

Convex surface $=2\pi ab$.

Whole surface $=2\pi b^2+2\pi ab$.

(5°). Let $a=$ altitude, and $b=$ radius of base of cone.

Solidity of cone $=\frac{1}{3}\pi b^2a$.

Convex surface* $=\pi b\sqrt{b^2+a^2}$.

Whole surface $=\pi b^2+\pi b\sqrt{b^2+a^2}$.

* The surface of a cone when unwrapped becomes the sector of a

Examples.

(1) Let $u = x^3 - 6x^2 + 11x - 6$; find the values of x which make u a maximum or minimum.

$$\frac{du}{dx} = 3x^2 - 12x + 11 = 0;$$

$$\therefore\ x^2 - 4x + 4 = \tfrac{1}{3};\quad \therefore\ x = 2 \pm \frac{1}{\sqrt{3}} = 2 \pm \frac{\sqrt{3}}{3},$$

$$\frac{d^2u}{dx^2} = 6x - 12.$$

Let $x = 2 + \dfrac{\sqrt{3}}{3}$; $\therefore\ \dfrac{d^2u}{du^2} = 2\sqrt{3}$ indicates a minimum,

$x = 2 - \dfrac{\sqrt{3}}{3}$; $\therefore\ \dfrac{d^2u}{dx^2} = -2\sqrt{3}$ .... a maximum.

(2) Let $y = x\tan\theta - \dfrac{x^2}{4h\cos^2\theta}$; find x that y may be a maximum or minimum.

$$\frac{dy}{dx} = \tan\theta - \frac{x}{2h\cos^2\theta};\quad \frac{d^2y}{dx^2} = -\frac{1}{2h\cos^2\theta}.$$

From $\dfrac{dy}{dx} = 0$, $x = 2h\tan\theta\cos^2\theta = 2h\sin\theta\cos\theta$;

also $\dfrac{d^2y}{dx^2}$ is negative; $\therefore\ y$ is a maximum,

$$\text{and } y = 2h\tan\theta\,.\sin\theta\cos\theta - \frac{4h^2\sin^2\theta\cos^2\theta}{4h\cos^2\theta}$$

$$= 2h\sin^2\theta - h\sin^2\theta = h\sin^2\theta.$$

This is the equation to the path of a projectile, and the maximum value of y is the greatest height above the horizontal plane.

(3) $u = (\sin x)^m.\{\sin(a - x)\}^n$; find x that u may be a maximum or minimum.

circle, of which the centre is the vertex of the cone, and radius the slant side, and arc the circumference of the base of the cone.

But sector $= \tfrac{1}{2}$ rad $\times$ arc; and radius $= \sqrt{b^2 + a^2}$; arc $= 2\pi b$;

$\therefore$ Convex surface $= \pi b\sqrt{b^2 + a^2}$.

$$\frac{du}{dx} = m(\sin x)^{m-1} \sin(a-x)^n . \cos x$$
$$- n(\sin x)^m . \sin(a-x)^{n-1} \cos(a-x) = 0;$$
$$\therefore\ m \sin(a-x) . \cos x - n \sin x \cos(a-x) = 0;$$
$$\therefore\ \frac{\sin(a-x) . \cos x}{\cos(a-x) . \sin x} = \frac{n}{m};$$
$$\therefore\ \frac{\sin(a-x)\cos x - \cos(a-x)\sin x}{\sin(a-x)\cos x + \sin x \cos(a-x)} = \frac{n-m}{n+m};$$
$$\therefore\ \frac{\sin(a-2x)}{\sin a} = \frac{n-m}{n+m};$$

$a - 2x$ and $\therefore x$ may be found from the tables.

(4) $u = \dfrac{\log x}{x}$; find x that u may be a maximum.

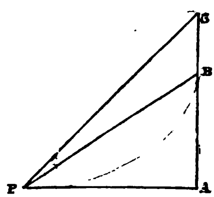
$$\frac{du}{dx} = \frac{1 - \log x}{x^2} = 0; \quad \therefore\ \log x = 1 = \log e;$$
$$\therefore\ x = e, \text{ and } u = \frac{1}{e}.$$

(5) Find that fraction which exceeds its second power by the greatest possible number.

Let x be the fraction;

$$\therefore\ u = x - x^2 \text{ is a maximum};$$
$$\therefore\ \frac{du}{dx} = 1 - 2x = 0; \quad \therefore\ x = \tfrac{1}{2},$$
$$\frac{d^2u}{dx^2} = -2, \text{ or } x = \tfrac{1}{2}, \text{ is a maximum.}$$

(6) Find the distance of P from A, when $\angle\, CPB$ is a maximum.



$AB = a$, $\quad AP = x$,
$AC = b$, $\quad \angle\, CPB = \theta$;

$$\therefore\ \theta = \angle\, CPA - \angle\, BPA$$
$$= \tan^{-1}\frac{b}{x} - \tan^{-1}\frac{a}{x},$$
$$\therefore\ \frac{d\theta}{dx} = -\frac{\frac{b}{x^2}}{1 + \frac{b^2}{x^2}} + \frac{\frac{a}{x^2}}{1 + \frac{a^2}{x^2}} = \frac{a}{x^2 + a^2} - \frac{b}{x^2 + b^2} = 0;$$

$$\therefore\ (a-b)\,x^2 = a^2b - ab^2; \quad \therefore\ x = \sqrt{ab};$$

$\therefore$ AP touches the circle circumscribing the $\triangle PBC$.

(7) Of all triangles upon the same base, and having the same perimeter, the isosceles has the greatest area;

$2P$ the perimeter and a the given base,

x and y the remaining sides;

$$\therefore\ \text{area} = \sqrt{P.(P-a).(P-x).(P-y)};$$

$\because$ P and $P-a$ are constant, and if $\sqrt{u}$ be a maximum, u is also a maximum.

Let $u = (P-x).(P-y) = (P-x).(a+x-P)$;

$$\therefore\ \frac{du}{dx} = -(a+x-P) + P - x = 0; \quad \therefore\ x = P - \frac{a}{2}.$$

$$\therefore\ y = 2P-(a+x) = 2P - \left(P+\frac{a}{2}\right) = P - \frac{a}{2} = x,$$

or the triangle is isosceles; area $= \frac{a}{2}\sqrt{P(P-a)}$.

(8) Divide a number a into two such parts, that the product of the m^{th} power of the one into the n^{th} power of the other may be a maximum.

x one part; $a-x$ the other; $u = x^m.(a-x)^n$;

$$\therefore\ \frac{du}{dx} = mx^{m-1}.(a-x)^n - x^m n.(a-x)^{n-1}$$

$$= x^{m-1}.(a-x)^{n-1}\{ma-(m+n).x\} = 0,$$

whence $x=0$, $x=a$, and $x = \frac{ma}{m+n}$;

$$\frac{d^2u}{dx^2} = \{(m-1).x^{m-2}.(a-x)^{n-1} - (n-1).x^{m-1}.(a-x)^{n-2}\}$$

$$\{ma-(m+n).x\} - (m+n).x^{m-1}.(a-x)^{n-1},$$

which vanishes when $x=0$ and $x=a$, but if $x = \frac{ma}{m+n}$,

$$\frac{d^2u}{dx^2} = -(m+n).\left(\frac{ma}{m+n}\right)^{m-1}.\left(\frac{na}{m+n}\right)^{n-1};$$

$$\therefore\ x = \frac{ma}{m+n} \text{ gives } u = \text{maximum}.$$

$x=0$ and $x=a$ will give no results unless m and n are even.

And then $\frac{d^m u}{dx^m} = m.(m-1)(m-2)\ldots 2.1.(a-x)^n + \phi(x)$,

$$\frac{d^n u}{dx^n} = n.(n-1)(n-2)\ldots 2.1.x^m + \phi(a-x),$$

$$\text{and } \frac{d^m u}{dx^m} = m.(m-1)(m-2)\ldots 2.1.a^n, \text{ when } x = 0,$$

$$\text{and } \frac{d^n u}{dx^n} = n.(n-1)(n-2)\ldots 2.1.a^m, \text{ when } x = a;$$

both of which correspond to minima.

(9) $u^3 - 3aux + x^3 = 0$; find x when u is a maximum.

$$\therefore \frac{du}{dx}.(u^2 - ax) - au + x^2 = 0.$$

$$\text{But } \frac{du}{dx} = 0; \quad \therefore x^2 - au = 0, \text{ or } u = \frac{x^2}{a}.$$

Substitute in the original equation,

$$\frac{x^6}{a^3} - 3x^3 + x^3 = 0\ldots(1); \quad \therefore x^3 = 2a^3; \quad \therefore x = a.\sqrt[3]{2}.$$

Differentiating a second time,

$$\frac{d^2u}{dx^2}(u^2 - ax) + \frac{du}{dx}.(2u\frac{du}{dx} - a) - a\frac{du}{dx} + 2x = 0.$$

$$\text{But } \frac{du}{dx} = 0, \text{ and } u^2 - ax = \frac{x^4}{a^2} - ax = \frac{x}{a^2}(x^3 - a^3) = ax;$$

$$\therefore \frac{d^2u}{dx^2} = \frac{-2x}{ax} = -\frac{2}{a},$$

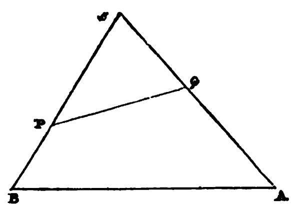
whence $x = a\sqrt[3]{2}$, gives $u = a\sqrt[3]{4}$, a maximum.

From (1) $x = 0$; $\therefore u = 0$; $\dfrac{d^2u}{dx^2} = \dfrac{-2x}{u^2 - ax} = \dfrac{0}{0}$; if $x = 0$.

Treating the fraction as a vanishing one,

$$\frac{d^2u}{dx^2} = \frac{-2}{2u\frac{du}{dx} - a} = \frac{2}{a}, \text{ if } x = 0; \quad \therefore u = 0, \text{ is a minimum.}$$

(10) Bisect a triangle by the shortest line.



ABC the triangle, and PQ the shortest line.

$$\left.\begin{array}{l} CP = x \\ CQ = y \\ PQ = u \end{array}\right\} \begin{array}{l} a, b, c \text{ the three sides} \\ \text{of the triangle, } C \\ \text{the } \angle BCA. \end{array}$$

Then $\because \triangle ABC = 2\triangle CPQ$;

$$\therefore \frac{ab \sin C}{2} = 2.\frac{xy \sin C}{2} = xy \sin C; \quad \therefore ab = 2xy,$$

$$u^2 = x^2 + y^2 - 2xy \cos C = x^2 + \frac{a^2b^2}{4x^2} - ab \cos C = \text{mimimum};$$

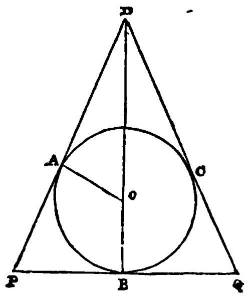
$$\therefore\ 2u\frac{du}{dx}=2x-\frac{a^2b^2}{2x^3}=0;$$

$$\therefore\ x^4=\frac{a^2b^2}{4},\ \text{or}\ x=\sqrt{\frac{ab}{2}},\ \text{and}\ y=\frac{ab}{2x}=\sqrt{\frac{ab}{2}};$$

$$\therefore\ u^2=\frac{ab}{2}+\frac{ab}{2}-ab\cos C=ab.(1-\cos C)=\frac{c^2-(a-b)^2}{2};$$

$$\therefore\ u=\sqrt{\frac{(c-a+b)(c+a-b)}{2}}.$$

(11) Describe about a given circle ABC, the least isosceles triangle.



DPQ the triangle.

$DO=x$; $OA=a$; $\therefore\ DA=\sqrt{x^2-a^2}$.

Now $PB=\frac{DB}{DA}OA=\frac{(x+a)\times a}{\sqrt{x^2-a^2}}$;

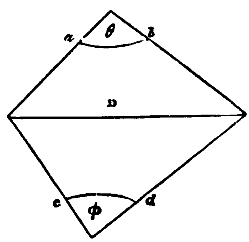
$$\therefore\ \triangle DPQ=PB\times DB$$

$$=\frac{a(x+a)}{\sqrt{x^2-a^2}}\times(x+a)$$

$$=a.\frac{(x+a)^{\frac{3}{2}}}{\sqrt{x-a}}=\text{minimum}.$$

Whence, if $u=\frac{(x+a)^3}{x-a}$, $x=2a$, and $\triangle=a^2.3\sqrt{3}$.

(12) Find the greatest area included by four straight lines.



Let a, b, c, d be the four lines,

θ the $\angle$ included by a, b,

ϕ ... $\angle$ .... c, d,

D the diagonal;

$$\therefore\ u=\text{area}=\frac{ab.\sin\theta}{2}$$

$$+\frac{cd.\sin\phi}{2}=\text{maximum};$$

$$\therefore\ \frac{du}{d\theta}=\tfrac{1}{2}.\left(ab.\cos\theta+cd.\cos\phi.\frac{d\phi}{d\theta}\right)=0.$$

But $c^2+d^2-2cd\cos\phi=D^2=a^2+b^2-2ab.\cos\theta$;

$$\therefore\ cd.\frac{d\phi}{d\theta}=ab.\frac{\sin\theta}{\sin\phi};$$

$\therefore$ substituting this value, and dividing by $\dfrac{ab}{2}$,

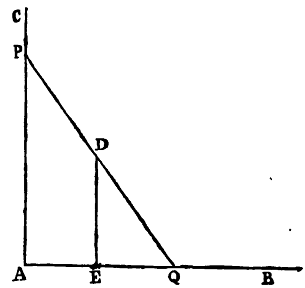
$$\cos\theta + \cos\phi \cdot \frac{\sin\theta}{\sin\phi} = 0;$$

$$\therefore \sin\phi \cdot \cos\theta + \sin\theta \cdot \cos\phi = \sin(\phi + \theta) = 0 = \sin\pi;$$

$$\therefore \phi + \theta = \pi,$$

or the quadrilateral may be inscribed in a circle.

(13) Through a given point D to draw the shortest line PQ terminated by two given straight lines AB, AC; the angle A being a right angle.



$AQ = x$; $AE = a$; $u = PQ$,
$AP = y$; $ED = b$;

$\therefore u = \sqrt{x^2 + y^2}$, a minimum ..(1),

and $\dfrac{a}{x} + \dfrac{b}{y} = 1$; which is the equation to PQ...(2).

From (1) $xdx + ydy = 0$; from (2) $\dfrac{adx}{x^2} + \dfrac{bdy}{y^2} = 0$;

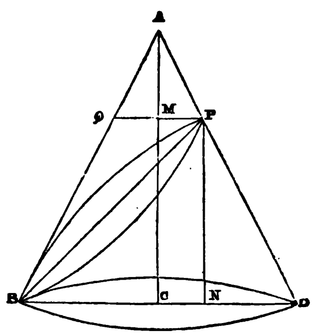
$$\therefore dx = -\frac{y}{x}dy; \quad \therefore \frac{a}{x^2} \cdot \frac{y}{x} - \frac{b}{y^2} = 0; \quad y = x\sqrt[3]{\frac{b}{a}};$$

$$\therefore \frac{a}{x} + \frac{\sqrt[3]{ab^2}}{x} = 1; \quad \therefore x = \sqrt[3]{a} \cdot (a^{\frac{2}{3}} + b^{\frac{2}{3}});$$

$$\therefore y = \sqrt[3]{b}\,(a^{\frac{2}{3}} + b^{\frac{2}{3}}); \quad \therefore u^2 = (a^{\frac{2}{3}} + b^{\frac{2}{3}})(a^{\frac{2}{3}} + b^{\frac{2}{3}})^2;$$

$$\therefore u^{\frac{2}{3}} = a^{\frac{2}{3}} + b^{\frac{2}{3}}.$$

(14) Cut the greatest ellipse from a given cone. ABD the cone. PB the elliptic section.



$AC = a$, $CN = x$,
$BC = \beta$, $NP = y$,

PB the axis-major $= 2a$,
and axis-minor $= 2b$.

Now area of ellipse $= \pi ab$.

And $2b = \sqrt{PQ \times BD}$
$= \sqrt{2x \times 2\beta} = 2\sqrt{\beta x}$,

$$2a = \sqrt{BN^2 + NP^2} = \sqrt{(\beta + x)^2 + NP^2}.$$

But $NP = CA \times \frac{DN}{CD} = a \,.\, \frac{\beta - x}{\beta}$;

$$\therefore\ 2a = \sqrt{(\beta + x)^2 + \frac{a^2}{\beta^2}(\beta - x)^2};$$

$$\therefore\ \text{area} = \pi \frac{\sqrt{\beta x}}{2} \,.\, \sqrt{(\beta + x)^2 + \frac{a^2}{\beta^2}(\beta - x)^2};$$

$$\therefore\ u = x\{(\beta + x)^2 + \frac{a^2}{\beta^2} \,.\, (\beta - x)^2\},\ \text{a maximum};$$

$$\therefore\ (\beta + x)^2 + \frac{a^2}{\beta^2}(\beta - x)^2 + 2x\{\beta + x - \frac{a^2}{\beta^2}(\beta - x)\} = 0;$$

$$\text{whence}\ \frac{3(a^2 + \beta^2)x^2}{\beta^2} - \frac{4(a^2 - \beta^2)x}{\beta} = -(a^2 + \beta^2);$$

$$\therefore\ x^2 - \frac{4}{3}\frac{(a^2 - \beta^2)}{(a^2 + \beta^2)}\beta x = -\frac{\beta^2}{3};$$

$$\therefore\ x = \frac{2}{3} \,.\, \beta \,.\, \frac{(a^2 - \beta^2)}{a^2 + \beta^2} \pm \sqrt{\frac{4}{9}\beta^2 \,.\, \left(\frac{a^2 - \beta^2}{a^2 + \beta^2}\right)^2 - \frac{\beta^2}{3}}$$

$$= \frac{2\beta(a^2 - \beta^2) \pm \beta\sqrt{a^4 - 14\beta^2a^2 + \beta^4}}{3(a^2 + \beta^2)};$$

and the problem is possible if $a^4 - 14\beta^2a^2 + \beta^4$ is positive. The limit of possibility is when the surd disappears.

$$\text{Then}\ a^4 - 14\beta^2a^2 + 49\beta^4 = 48\beta^4;$$

$$\therefore\ a^2 = 7\beta^2 \pm \sqrt{48\beta^4} = \beta^2\{7 \pm 4\sqrt{3}\};$$

$$\therefore\ a = \beta(2 \pm \sqrt{3}),$$

$$\text{and}\ x = \frac{2\beta}{3} \,.\, \frac{6 \pm 4\sqrt{3}}{8 \pm 4\sqrt{3}} = \frac{\beta}{3} \,.\, \frac{3 + 2\sqrt{3}}{2 + \sqrt{3}} = \frac{\beta}{\sqrt{3}}.$$

(15) The content of a cone being given, find its form when its surface is a maximum.

x the altitude, and y the radius of the base.

Let $\frac{\pi a^3}{3}$ be the given content $= \therefore\ \frac{\pi y^2 x}{3}$.

Then u = surface = convex surface + base;

$$\therefore\ u = \pi y\sqrt{x^2 + y^2} + \pi y^2.$$

$$\text{But } y^2 = \frac{a^3}{x}; \quad \therefore\ y^2 + x^2 = \frac{a^3 + x^3}{x};$$

$$\therefore\ u = \pi a^{\frac{3}{2}} \left\{ \frac{\sqrt{a^3 + x^3} + a^{\frac{3}{2}}}{x} \right\};$$

whence because $\dfrac{du}{dx} = 0$; $\quad x^3 - 2a^3 = 2a^{\frac{3}{2}} \sqrt{x^3 + a^3}$;

$$\therefore\ x = 2a; \quad y^2 = \frac{a^3}{x} = \frac{a^2}{2}, \quad \text{or } y = \frac{a}{\sqrt{2}},$$

$$\text{and } u = \frac{\pi a}{\sqrt{2}} \cdot \sqrt{4a^2 + \frac{a^2}{2}} + \frac{\pi a^2}{2} = 2\pi a^2.$$

Examples.

(1) Let $u = x^3 - 7x^2 + 8x + 32$.

$x = 4$; $\quad u = 16$ a minimum;

$x = \frac{2}{3}$; $\quad u = 34\frac{14}{27}$ a maximum.

(2) $u = x^3 - 3x^2 - 9x + 30$.

$x = -1$, gives $u = 35$ a maximum;

$x = 3$, gives $u = 3$ a minimum.

(3) $u = \sin^3 x \cdot \cos x$; $\quad x = 60$; $\quad u =$ a maximum.

(4) Divide a number into two such parts, that their product multiplied by the difference of their squares shall be a maximum.

$2a$ the number, $a + x$ and $a - x$ the parts;

$\therefore\ u = (a^2 - x^2) \cdot 4ax =$ maximum, whence $x = \dfrac{a}{\sqrt{3}}$.

(5) Divide a number a into two such factors, that the sum of their squares shall be a minimum; $\quad x = \sqrt{a}$.

(6) $u = x^{\frac{1}{x}}$; $x = e$; $u = e^{\frac{1}{e}}$ a maximum.

(7) Into how many equal parts must a number a be divided that their product may be the greatest?

$$x = \frac{a}{e}; \quad u = e^{\frac{a}{e}}.$$

(8) Let $u=(mx+n).(ny+m)$ be a maximum, and $a^{mx}.b^{ny}=c$; find x.

$$x=\frac{\log\left(\frac{cb^m}{a^n}\right)}{\log(a^{2m})}.$$

(9) $u=e^{-ax}-e^{-bx}$, a minimum: $a>b$, and shew it is a minimum.

(10) $u=\dfrac{\sin x}{1+\tan x}=$ a maximum; if $x=45^\circ$.

(11) $u=\dfrac{(\tan x)^3}{\tan 3x}=$ a minimum; if $x=22^\circ.30'$.

(12) $u=\sec x.(a-b\tan x)^2$ a maximum or minimum.

$$\tan x=\frac{1}{6b}(a\pm\sqrt{a^2-24b^2}),$$

the upper sign gives a maximum; the lower a minimum.

(13) $u=3x^4-28ax^3+84a^2x^2-96a^3x+48b^4$.

$x=a$; $u=48b^4-37a^4$ a minimum;

$x=2a$; $u=48b^4-32a^4$ a maximum;

$x=4a$; $u=48b^4-64a^4$ a minimum.

(14) $u=x(a-x)^2(2a-x)^3$.

$x=\dfrac{a}{6}(5\quad\sqrt{13})$; u is a maximum;

$x=a$; u is a minimum;

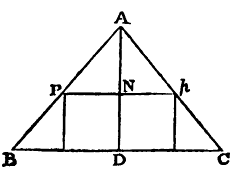
$x=\dfrac{a}{6}(5+\sqrt{13})$; u is a maximum.

(15) $au^3-u^2x^2+x^4=0$; $u=$ a minimum; find x.

$$x=2a\sqrt{2};\quad u=4a.$$

(16) $u=ax-x^2.\sin\dfrac{a}{x}.\cos\dfrac{a}{x}$ a maximum if $\pi x=2a$.

(17) Inscribe the greatest rectangle in a given triangle.



$AD=a$, $BC=b$, $AN=x$; $\therefore Pp=\dfrac{bx}{a}$;

$u=\dfrac{bx}{a}(a-x)$; $\therefore x=\dfrac{a}{2}$.

(18) Inscribe the greatest isosceles triangle in a given circle.

Let a = radius, the triangle is equilateral, side $= a\sqrt{3}$, area $= \dfrac{a^2 3\sqrt{3}}{4}$.

(19) Inscribe the greatest parallelogram within a given triangle ABC, A being one of the angles of the parallelogram. $AE = \frac{1}{2} AB$, then AE is one of the sides.

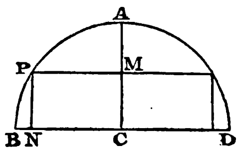
(20) Of all equiangular and isoperimetrical parallelograms, the equilateral has the greatest area.

(21) Of all triangles on the same base, and having equal vertical angles, the isosceles has the greatest perimeter.

(22) Given the base and vertical angle of a triangle, shew that when it is isosceles its area is a maximum.

(23) Of all triangles on the same base and having the same area, the isosceles has the least perimeter.

(24) Inscribe the greatest rectangle in a semicircle.



$CN = x$, $CA = a$, $NP = \sqrt{a^2 - x^2}$;

$\therefore\ u = 2PM \cdot CM = 2x\sqrt{a^2 - x^2}$;

$$\therefore\ x = \frac{a}{\sqrt{2}}, \text{ and } u = a^2$$

(25) The same construction applies to any curve.

$$AC = b,\ AM = x;\ \therefore\ PM = f(x), \text{ and } u = 2(b - x) \cdot f(x).$$

If BAD be a parabola; $u = 4(b - x)\sqrt{mx}$.

(26) If BAD be a circular segment;

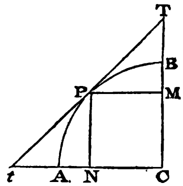
$$\therefore\ u = 2(b - x)\sqrt{2ax - x^2}.$$

(27) Given the difference of the angles at the base and the radius of the inscribed circle, find when the perimeter is greatest.

(28) If A be the vertex, S the focus and P a point in a parabola, find the value of the ratio of $AP : SP$, when it is greatest. Ratio $= \dfrac{2}{\sqrt{3}}$.

(29) Cut the greatest parabola from a given cone.

(30) Required the least triangle TCt which can be described about a given quadrant.



$$CA = a, \quad CM = x, \quad CN = y;$$

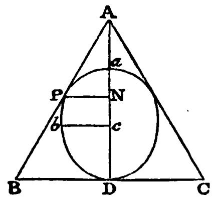
$$u = \tfrac{1}{2}\, CT \,.\, Ct; \quad CT = \frac{a^2}{x}; \quad Ct = \frac{a^2}{y};$$

and if $u =$ maximum, $x = y$ and $\angle ACP = 45^\circ$.

(31) Let APB be a parabolic arc and C the focus.

$$AN = x, \quad AC = a, \quad u = \frac{(x+a)^2 \sqrt{a}}{2\sqrt{x}}, \text{ whence } x = \frac{a}{3}.$$

(32) Inscribe the greatest ellipse in a given isosceles triangle.



Let $Da = 2x$, $cb = y$; $\therefore\ u = \pi \,.\, yx$.

$AD = a$; $DB = b$.

$$\text{Now } cN = \frac{ca^2}{cA} = \frac{x^2}{a-x};$$

$$\therefore\ aN = \frac{ax - 2x^2}{a-x}, \quad DN = \frac{ax}{a-x}.$$

$$\text{But } \frac{BD^2}{AD^2} \,.\, AN^2 = PN^2 = \frac{y^2}{x^2}(Na \,.\, ND);$$

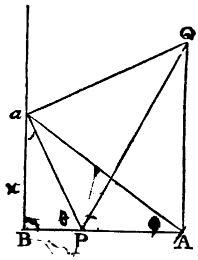
$$\therefore\ b^2 \left(\frac{a-2x}{a-x}\right)^2 = y^2 \,.\, \frac{(a-2x)\,a}{(a-x)^2}; \quad \therefore\ y^2 = \frac{b^2}{a}(a-2x);$$

$$\therefore\ u = \pi y x = \frac{\pi b}{\sqrt{a}} \,.\, x\sqrt{a-2x}; \quad \therefore\ x = \frac{a}{3}.$$

(33) Inscribe the greatest parabola in a given isosceles triangle. Axis $\frac{3}{4}$th of altitude of triangle.

(34) Within a given parabola inscribe the greatest parabola, the vertex of the latter being at the bisection of the base of the former. Axis = $\frac{2}{3}$ of given axis.

(35) Describe the greatest isosceles triangle about an ellipse, the major axis, and altitude being coincident.



(36) The corner of a leaf is turned back, so as just to reach the other edge of the page: find when the length of the crease is a minimum.

$$AP = x,\quad AB = a\ ;\quad \therefore\ Aa = \sqrt{2ax},$$

$$\text{also } Aa\,.\,PQ = 2AQ\,.\,AP\ ;$$

$$\therefore\ u^2 = PQ^2 = \frac{2x^3}{2x-a}\ ;\quad \therefore\ x = \frac{3a}{4}.$$

(37) The part turned down is least, if $x = \frac{2}{3}a$.

(38) Inscribe the greatest cylinder within a cone.

a, altitude of cone; x, of cylinder; $x = \dfrac{a}{3}$; $u = \dfrac{4\pi b^2 a}{27}$.

(39) Inscribe the greatest cone within a sphere.

$$x = \text{altitude of cone} = \frac{4a}{3}\ ;\quad u = \frac{32}{81}\,.\,\pi a^3.$$

(40) Given the surface of a cylinder, find its form that its volume may be a maximum. Altitude = diameter of base.

(41) Given the volume, find when the surface is least. Altitude = diameter of base.

(42) In the trapezium $ABCD$, the base $AB = a$, $AD = BC = b$, find CD, CD being parallel to AB, that the area may be a maximum; $2CD = \sqrt{8b^2 + a^2} + a$.

(43) PQ is a chord in a semicircle parallel to the diameter AB, join AQ, BP cutting in O: find AP that the triangle POQ may be the greatest possible. $AP = 38^\circ\,.\,40'$.

(44) Through a given point D between two given straight lines AB, AC (page 76): to draw PQ, so that $AP + AQ$ may be a minimum.

$$u = (\sqrt{a} + \sqrt{b})^2.$$

(45) Draw PQ so that PAQ may be least.

$$x = 2a\ ;\quad y = 2b\ ;\quad u = 2ab.$$

(46) ACB is a quadrant, C the centre, CB the horizontal radius is bisected in D, a point P is taken in the arc, and CP, PD are joined, shew that the angle CPD is greatest when PD is vertical.

(47) Find the vertical angle of the greatest right cone

which can be described by a right-angled triangle of given hypothenuse.

(48) The centres of two spheres (radii $r_1 r_2$) are at the extremities of a line $2a$, on which a circle is described. Find a point in the circumference, from which the greatest portion of spherical surface is visible.

If x and y be the distances of the point from the centres of the two spheres,

$$u = 2\pi r_1\left(r_1 - \frac{r_1^2}{x}\right) + 2\pi r_2\left(r_2 - \frac{r_2^2}{y}\right),$$

$$\text{whence } x = \frac{2ar_1}{\sqrt{r_1^2 + r_2^2}}.$$

(49) Find the position of a line passing through one of three given points, so that the rectangle of the perpendiculars from the other two points may be a maximum or minimum.

There are two lines respectively perpendicular, fulfilling the conditions.

(50) In a spherical triangle, find δ when $u = \theta - \phi$ is a minimum, and

$$\cos\theta = -\frac{\sin a + \sin\delta . \sin l}{\cos\delta . \cos l},$$

$$\cos\phi = -\frac{\sin\delta . \sin l}{\cos\delta . \cos l},$$

differentiating the three equations and eliminating $d\theta$ and $d\phi$,

$$\frac{\sin\theta}{\sin\phi} = \frac{\sin a}{\sin l}\sin\delta + 1;$$

$$\text{whence } \sin\delta = -\sin l . \tan(\tfrac{1}{2}a).$$

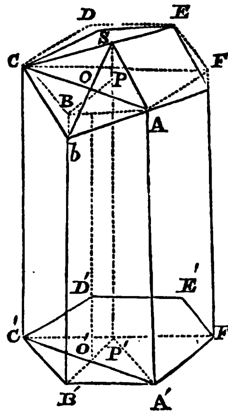
This is the problem of the shortest twilight, and if Z be the zenith, P the north pole, S the sun when twilight begins, s when it ends, $\theta = ZPS$, $\phi = ZPs$; $\therefore\ \theta - \phi$ converted into time is the duration; δ is the sun's declination, l the latitude of the place, a is the depression of S below the horizon, generally taken $= 18^\circ$; $\therefore\ \sin\delta = -\sin l \tan 9^\circ$; the negative sign shews the sun is on the south side of the equator.

The Cells of Bees.

93. The first examination of the comb of the bee-hive presents a collection of waxen cells, the upper surface being parallel to the lower: but these parallel surfaces being separated, each cell is found to be of a prismatic form; its base being a regular hexagon, and the other end of the prism formed of three equal rhombuses, composing the solid angle S. And it is remarkable that the two collections of cells, which by their junction form the comb, are so united that the axis of any one cell of one collection is in the continuation of the line of junction of three other cells of the other collection.

That the bases should be regular hexagons is an instance of the economy of nature; for only three figures, triangles, squares, and hexagons can completely occupy space; and of these, including the same area, the hexagon has the least perimeter.

Prop. Join A and C the extremities of $A'A$, $C'C$ of two equal edges of a prism, and let a plane through CA parallel to the base meet the axis in P; and let a plane inclined to the base, also through CA cut the axis in S and $B'B$ in b. Then



$SP = Bb$, $\triangle AOP = \triangle AOB$;

$\therefore$ pyramid $ACSP =$ pyramid $ACBb$.

Hence whatever may be the inclination SOP of the plane SCb to the base of the cell, the solid content of the cell remains unaltered, but the surface varies with $\angle SOP$.

To find $\angle SOP$ when the surface of the cell is a minimum.

Let $\angle SOP = \theta$, $A'B' = AB = a$, $AA' = b$.

$$\text{Then } BO = \frac{a}{2};\ Ob = OS = \frac{BO}{\cos\theta} = \frac{a}{2\cos\theta}.$$

$$Bb = SP = OB\tan\theta = \frac{a}{2}\tan\theta;$$

$$\therefore \text{ trapezium } AA'B'b = BA' - \triangle ABb = ba - \frac{a^2}{4}\tan\theta;$$

Note. BP in the figure ought to pass through O.

$$\therefore \text{ lateral surface of cell} = 3a\left(2b - \frac{a}{2}\tan\theta\right).$$

$$\text{And rhombus} = SO \times AC = SO \times A'C' = \frac{a}{2\cos\theta} . a\sqrt{3};$$

$$\therefore \text{ surface of the tetrahedron} = \frac{3a^2\sqrt{3}}{2\cos\theta};$$

$$\therefore u = 3a\left\{2b - \frac{a}{2}\tan\theta + \frac{a\sqrt{3}}{2\cos\theta}\right\};$$

$$\therefore \frac{du}{d\theta} = 3a\left\{-\frac{a}{2}\sec^2\theta + \frac{a}{2}\sqrt{3} . \sec\theta . \tan\theta\right\} = 0;$$

$$\therefore \sec\theta = \sqrt{3}\tan\theta; \ \therefore \sin\theta = \frac{1}{\sqrt{3}}; \ \therefore \theta = 35^\circ . 15' . 51''.$$

$$\text{Also } Ob = \frac{a}{2}\sqrt{\frac{3}{2}}; \ Cb = \sqrt{(CO)^2 + (Ob)^2} = \frac{3a}{2\sqrt{2}}.$$

$$\tan OCb = \frac{Ob}{CO} = \frac{1}{\sqrt{2}}; \ \therefore \sin OCb = \frac{1}{\sqrt{3}};$$

$$\therefore \angle OCb = \angle SOP,$$

$$\text{and } \angle SCb = 2\angle SOP.$$

Hence the acute angle of the rhomb is double the inclination of the rhomb to the base of the cell. These results agree with the most exact measurements made in a multitude of cells*.

94. If $x = a$, make $\frac{du}{dx} = \infty$, the preceding rules are inapplicable, since they are founded on the supposition that $f(a+h)$ is expanded according to the ascending integral powers of h by means of Taylor's Theorem; but which is not the case, when the differential coefficients become infinite.

Let then $f(a+h)$ be expanded by the ordinary methods, and assume

$$f(a+h) = f(a) + Ph^\alpha + Qh^\beta + Rh^\gamma + \&c.$$

where α is the least of all the indices of h;

$$\therefore f(a+h) - f(a) = Ph^\alpha + Qh^\beta + Rh^\gamma + \&c.\ldots(1),$$

$$\text{and } f(a-h) - f(a) = P(-h)^\alpha + Q(-h)^\beta + \&c.$$

by writing $-h$ for h in series (1).

* Puissant's Geometry: Cresswell's Maxima and Minima.

Now if h be made very small, the algebraical sign of the developements will depend on that of their first term. If therefore there should be a maximum or minimum, since $f(a+h)-f(a)$, and $f(a-h)-f(a)$ must have the same signs, Ph^α and $P(-h)^\alpha$; and $\therefore$ h^α and $(-h)^\alpha$ must have the same sign, or α must either be an even number or a fraction with an even number for its numerator.

(1) If α be an even number, it shews that at a maximum or minimum the first existent term of the developement must involve an even power of h, a conclusion we have already come to in the preceding pages.

(2) If α be a fraction, it must be of the form $\dfrac{2n}{2m+1}$.

Ex. Let $u = b + c(x-a)^{\frac{2}{3}}$.

Here $\dfrac{du}{dx} = \dfrac{2c}{3}\dfrac{1}{(x-a)^{\frac{1}{3}}}$, which is infinite, if $x = a$.

But $x = a$, gives $u = b$,

$x = a + h$ gives $u = b + ch^{\frac{2}{3}}$,

$x = a - h$ ...... $u = b + ch^{\frac{2}{3}}$,

and $f(a+h)$ and $f(a-h)$ are both $> f(a)$, if c be positive, $< f(a)$, if c be negative.

If $\therefore$ c be positive, $x = a$ makes $u = b$ a minimum,

c be negative, $x = a$ .... $u = b$ a maximum.

CHAPTER VIII.

Functions of two or more Variables.—Implicit Functions.

95. As yet we have only treated of functions of a single variable; we next proceed to the case in which $u = f(xy)$, where x and y are independent of each other, and the value of u corresponding to new values $x + h$, and $y + k$, of x and y, is required.

Now when u is a function of x and y, u may vary on three suppositions; 1st, x may vary, and y remain constant; 2nd, y may vary, and x remain constant; and 3rd, x and y may both vary together.

Thus suppose $u = xy^2$, and let x become $x + h$, and y remain constant; therefore if u' be the value of u,

$$u' = (x + h)\, y^2 = xy^2 + y^2 h.$$

Next let y become $y + k$, and x be constant, and let u_1 be the value of u;

$$\therefore\ u_1 = x(y + k)^2 = xy^2 + 2xyk + xk^2.$$

Again, in the equation $u = xy^2$ write $x + h$ for x, and $y + k$ for y, and let u_2 be the value of u, or $u_2 = f(x + h,\ y + k)$;

$$\therefore\ u_2 = (x + h)(y + k)^2 = xy^2 + y^2 h + 2xyk + 2ykh + xk^2 + k^2 h,$$

the same result as would have been obtained had we put

$$y + k \text{ for } y \text{ in } u', \text{ or } x + h \text{ for } x \text{ in } u_1.$$

96. Next considering the question in a general point of view.

Let $u = f(x,\ y)$, then if y remain constant while x becomes $x + h$, we have, by Taylor's Theorem,

$$f(x+h,\ y) = u + \frac{du}{dx} h + \frac{d^2u}{dx^2} \frac{h^2}{1.2} + \frac{d^3u}{dx^3} \frac{h^3}{2.3} + \&c.;$$

or, if x remain constant while y becomes $y + k$,

$$f(x,\ y + k) = u + \frac{du}{dy} k + \frac{d^2u}{dy^2} \frac{k^2}{1.2} + \frac{d^3u}{dy^3} \cdot \frac{k^3}{2.3} + \&c.$$

Suppose now that x and y both vary; or x become $x + h$, and y become $y + k$; it is not possible to make both these assumptions at once: but if we use either of the two series, for $f(x + h,\ y)$ or $f(x,\ y + k)$, and in the former put $y + k$

for y, or in the latter $x+h$ for x, we shall in either case have $f(x+h,\ y+k)$, and its true developement.

Assuming the first expansion,

$$f(x+h\,.\,y)=u+\frac{du}{dx}.h+\frac{d^2u}{dx^2}\frac{h^2}{1.2}+\frac{d^3u}{dx^3}\frac{h^3}{2.3}+\&c.$$

But $u=f(xy)$, and therefore $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$, are also functions of x and y, if therefore y become $y+k$; u, $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$; &c. will become functions of $y+k$, and may be expanded by Taylor's Theorem, x being considered constant.

Let therefore y become $y+k$;

$$\therefore\ u \text{ becomes } u+\frac{du}{dy}.k+\frac{d^2u}{dy^2}\frac{k^2}{1.2}+\frac{d^3u}{dy^3}.\frac{k^3}{2.3}+\&c. \ldots\ldots (\alpha),$$

and to obtain the values of $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$, &c. we must write $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$, &c. for u in the series (α);

$$\therefore\ \frac{du}{dx} \text{ becomes } \frac{du}{dx}+\frac{d.\left(\frac{du}{dx}\right)}{dy}k+\frac{d^2.\left(\frac{du}{dx}\right)}{dy^2}.\frac{k^2}{1.2}+\&c.$$

$$\frac{d^2u}{dx^2}\ \ldots\ldots\ldots\ \frac{d^2u}{dx^2}+\frac{d.\left(\frac{d^2u}{dx^2}\right)}{dy}.k+\&c.$$

$$\frac{d^3u}{dx^3}\ \ldots\ldots\ldots\ \frac{d^3u}{dx^3}+\frac{d.\left(\frac{d^3u}{dx^3}\right)}{dy}.k+\&c.$$

But it has been agreed to write $\frac{d^2u}{dy\,.\,dx}$ for $\frac{d.\left(\frac{du}{dx}\right)}{dy}$, which expresses that the function has been differentiated twice, 1st considering x, and then y as variable;

and $\frac{d.\left(\frac{d^2u}{dx^2}\right)}{dy}$ is written $\frac{d^3u}{dydx^2}$, and $\frac{d^n.\left(\frac{d^mu}{dx^m}\right)}{dy^n}$ is written $\frac{d^{m+n}.u}{dy^n.dx^m}$, denoting the differential coefficient when the function has been differentiated m times with regard to x, and n times with regard to y.

Making these substitutions, and multiplying the expansion of $\frac{du}{dx}$ by h, that of $\frac{d^2u}{dx^2}$ by $\frac{h^2}{1.2}$, &c. we shall have

$$\begin{aligned} f(x+h,\ y+k) = u &+ \frac{du}{dy}k + \frac{d^2u}{dy^2}\frac{k^2}{1.2} + \frac{d^3u}{dy^3}\frac{k^3}{2.3} + \text{\&c.} \\ &+ \frac{du}{dx}h + \frac{d^2u}{dy.dx}hk + \frac{d^3u}{dy^2.dx}.\frac{hk^2}{1.2} + \text{\&c.} \\ &+ \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \frac{d^3u}{dy.dx^2}.\frac{h^2k}{1.2} + \text{\&c.} \\ &+ \frac{d^3u}{dx^3}\frac{h^3}{2.3} + \text{\&c.} \\ &+ \text{\&c.} \ldots\ldots + \text{\&c.} \end{aligned}$$

97. But this developement was obtained, by first supposing x, and then y to vary; but manifestly we should have had an equal result, had y first become $y+k$, and then x become $x+h$. On this supposition we have

$$f(x,\ y+k) = u + \frac{du}{dy}k + \frac{d^2u}{dy^2}\frac{k^2}{1.2} + \frac{d^3u}{dy^3}\frac{k^3}{2.3} + \text{\&c.}$$

put $x+h$ for x,

$$\therefore\ u \text{ becomes } u + \frac{du}{dx}h + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\frac{h^3}{2.3} + \text{\&c.},$$

$$\frac{du}{dy} \ldots\ldots\ldots\ldots \frac{du}{dy} + \frac{d^2u}{dxdy}h + \frac{d^3u}{dx^2dy}.\frac{h^2}{1.2} + \text{\&c.},$$

$$\frac{d^2u}{dy^2} \ldots\ldots\ldots\ldots\ldots\ldots \frac{d^2u}{dy^2} + \frac{d^3u}{dxdy^2}\frac{h}{1} + \text{\&c.},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \text{\&c.}:$$

whence by substitution the total developement becomes

$$\begin{aligned} f(x+h,\ y+k) = u &+ \frac{du}{dx}h + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\frac{h^3}{2.3} + \text{\&c.} \\ &+ \frac{du}{dy}k + \frac{d^2u}{dxdy}kh + \frac{d^3u}{dx^2dy}\frac{h^2k}{1.2} + \text{\&c.} \\ &+ \frac{d^2u}{dy^2}\frac{k^2}{1.2} + \frac{d^3u}{dxdy^2}\frac{k^2h}{1.2} + \text{\&c.} \\ &+ \frac{d^3u}{dy^3}\frac{k^3}{2.3} + \text{\&c.} \\ &+ \text{\&c.} \end{aligned}$$

COR. 1. Since the series are equal, the coefficients of the same powers of h and k ought to be equal;

$$\therefore \frac{d^2u}{dydx} = \frac{d^2u}{dxdy},$$

$$\frac{d^3u}{dy^2dx} = \frac{d^3u}{dxdy^2},$$

$$\frac{d^3u}{dydx^2} = \frac{d^3u}{dx^2dy},$$

$$\&c. = \&c.$$

$$\text{and } \frac{d^{m+n}.u}{dy^m dx^n} = \frac{d^{m+n}u}{dx^n dy^m}.$$

Hence the order of differentiation is indifferent.

COR. 2. Again, $\because \frac{d^2u}{dydx} = \frac{d^2u}{dxdy}$; $\therefore$ writing $\frac{du}{dx}$ for u,

$$\frac{d^2.\left(\frac{du}{dx}\right)}{dydx} = \frac{d^2.\left(\frac{du}{dx}\right)}{dxdy},$$

$$\text{or } \frac{d^3u}{dydx^2} = \frac{d^3u}{dxdydx},$$

$$\text{and } \frac{d^2.\left(\frac{du}{dy}\right)}{dydx} = \frac{d^2.\left(\frac{du}{dy}\right)}{dxdy};$$

$$\therefore \frac{d^3.u}{dydxdy} = \frac{d^3.u}{dxdy^2}.$$

98. Since $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$, $\frac{d^3u}{dx^3}$, &c. have been obtained by the consideration of x alone being the independent variable, such differential coefficients have been called *partial* differential coefficients, and for the same reason $\frac{du}{dy}$, $\frac{d^2u}{dy^2}$, &c. are also called partial differential coefficients, and these partial differential coefficients are frequently included within brackets, thus $\left(\frac{du}{dx}\right)$ is the partial differential coefficient with respect to x, and $\left(\frac{du}{dy}\right)$ is the partial differential coefficient with respect to y, and $\left(\frac{du}{dx}\right)dx$, and $\left(\frac{du}{dy}\right)dy$, are the partial differentials of u, with regard to x and y respectively.

99. The term $\frac{du}{dx}h+\frac{du}{dy}k$, which involves only the first powers of h and k is called the *total* differential of u, and putting dx for h, and dy for k, is thus written;

$$du=\left(\frac{du}{dx}\right).dx+\left(\frac{du}{dy}\right)dy,$$

or the total differential of $u=f(xy)$ is the sum of the partial differentials.

100. From the first differential of u, we may form by differentiation the successive differentials d^2u, d^3u; &c.

$$\text{For } du=\left(\frac{du}{dx}\right)dx+\left(\frac{du}{dy}\right)dy\ \ldots\ldots\ldots\ (\beta).$$

And differentiating, considering $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$ as functions of x and y, and dx and dy constant, we have, by writing successively, $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$ for u in (β),

$$d.\left(\frac{du}{dx}\right)=\left(\frac{d^2u}{dx^2}\right).dx+\frac{d^2u}{dydx}.dy,$$

$$d.\left(\frac{du}{dy}\right)=\frac{d^2u}{dxdy}.dx+\left(\frac{d^2u}{dy^2}\right)dy.$$

Then substituting these values, since

$$d^2u=d.\left(\frac{du}{dx}\right).dx+d\left(\frac{du}{dy}\right).dy,$$

$$d^2u=\frac{d^2u}{dx^2}.dx^2+2.\frac{d^2u}{dx.dy}.dy.dx+\frac{d^2u}{dy^2}.dy^2.$$

Again, to find d^3u, substituting as before

$$d.\left(\frac{d^2u}{dx^2}\right)=\left(\frac{d^3u}{dx^3}\right).dx+\frac{d^3u}{dydx^2}.dy,$$

$$d.\left(\frac{d^2u}{dxdy}\right)=\frac{d^3u}{dx^2dy}.dx+\frac{d^3u}{dydxdy}.dy,$$

$$d.\left(\frac{d^2u}{dy^2}\right)=\frac{d^3u}{dxdy^2}.dx+\left(\frac{d^3u}{dy^3}\right)dy;$$

$$\therefore\ d^3u=\left(\frac{d^3u}{dx^3}\right)dx^3+3.\frac{d^3u}{dx^2dy}dx^2dy$$

$$+3.\frac{d^3u}{dy^2dx}.dy^2.dx+\left(\frac{d^3u}{dy^3}\right)dy^3.$$

101. The law of continuity is almost obvious; for the numerical coefficients appear to be those of the terms of the expansion of the binomial $(h+k)^n$: but to prove this, let

$$d^n u = \frac{d^n u}{dx^n} dx^n + n \cdot \frac{d^n u}{dx^{n-1} dy} \cdot dx^{n-1} dy$$
$$+ n \cdot \frac{n-1}{2} \frac{d^n u}{dx^{n-2} dy^2} dx^{n-2} dy^2 + \&c.$$

Differentiating the successive terms by means of

$$du = \frac{du}{dx} dx + \frac{du}{dy} \cdot dy,$$

$$d \cdot \left(\frac{d^n u}{dx^n}\right) = \frac{d^{n+1} u}{dx^{n+1}} \cdot dx + \frac{d^{n+1} u}{dx^n dy} \cdot dy \ldots\ldots\ldots\ldots (1),$$

$$d \cdot \left(\frac{d^n u}{dx^{n-1} dy}\right) = \frac{d^{n+1} u}{dx^n dy} \cdot dx + \frac{d^{n+1} u}{dx^{n-1} dy^2} \cdot dy \ldots\ldots\ldots (2),$$

$$d \cdot \left(\frac{d^n u}{dx^{n-2} dy^2}\right) = \frac{d^{n+1} u}{dx^{n-1} dy^2} \cdot dx + \frac{d^{n+1} u}{dx^{n-2} dy^3} \cdot dy \ldots\ldots\ldots (3).$$

$$\&c. = \&c.$$

Multiply (1) by dx^n, (2) by $n \cdot dx^{n-1} dy$, (3) by

$n \cdot \left(\frac{n-1}{2}\right) \cdot dx^{n-2} dy^2$, and adding

$$d^{n+1} u = \frac{d^{n+1} u}{dx^{n+1}} \cdot dx^{n+1} + (n+1) \cdot \left(\frac{d^{n+1} u}{dx^n dy}\right) \cdot dx^n dy$$
$$+ \frac{(n+1) \cdot n}{2} \cdot \frac{d^{n+1} u}{dx^{n-1} dy^2} \cdot dx^{n-1} dy^2 + \&c.$$

or if the formula be true for the index n, it is true for $n+1$, but it is true when $n = 3$; it is $\therefore$ always true.

Cor. If instead of dx and dy we write h and k,

$$du = \frac{du}{dx} h + \frac{du}{dy} k,$$

$$d^2 u = \frac{d^2 u}{dx^2} h^2 + 2 \frac{d^2 u}{dy dx} hk + \frac{d^2 u}{dy^2} k^2,$$

$$d^3 u = \frac{d^3 u}{dx^3} h^3 + 3 \frac{d^3 u}{dx^2 dy} h^2 k + 3 \frac{d^3 u}{dy^2 dx} hk^2 + \frac{d^3 u}{dy^3} k^3,$$

&c.

$$\therefore\ u_2 = f\{(x+h),\ (y+k)\}$$
$$= u + du + \frac{d^2u}{1.2} + \frac{d^3u}{2.3} + \&c.$$
$$+ \frac{d^nu}{1.2.3\ldots n} + \&c.,$$

or the expansion of $f(x+h,\ y+k)$ may be found from the successive differentiation of $u = f(x,\ y)$.

102. Again, if $u = f(x, y, z)$, and if $x+h,\ y+k,\ z+m$, be new values of x, y, z, and u_2 the value of u,

$$u_2 = u + \frac{du}{dx}h + \frac{du}{dy}k + \frac{du}{dz}m$$
$$+ Ah^2 + Bk^2 + Cm^2 + \&c.$$

For, supposing z to be constant while x and y become $x+h$, and $y+k$ respectively;

$$\therefore f(x+h,\ y+k,\ z) = u + \frac{du}{dx}h + \frac{du}{dy}k + \&c.$$

Let z become $z+m$;

$$\therefore\ u \text{ becomes } u + \frac{du}{dz}m + \&c.$$

$$\frac{du}{dx} \cdots\cdots\cdots\cdots \frac{du}{dx} + \frac{d^2u}{dzdx}m + \&c.$$

$$\frac{du}{dy} \cdots\cdots\cdots\cdots \frac{du}{dy} + \frac{d^2u}{dzdy}m + \&c.$$

$$\&c. \quad = \quad \&c.$$

$$u_2 = f(x+h,\ y+k,\ z+m) = u + \frac{du}{dz}m + \frac{du}{dx}h + \frac{du}{dy}k + \&c.$$

and in a similar manner may the expansion of a function of four or more variables be effected.

Cor. Hence, if for m, h, and k, we put dz, dx; and dy,

$$du = d.f(x,\ y,\ z) = \left(\frac{du}{dx}\right)dx + \left(\frac{du}{dy}\right)dy + \left(\frac{du}{dz}\right)dz.$$

This result may however be obtained in the following manner:

103. Let $u = f(x,\ y,\ z)$; find du.

Let $n = \phi(y,\ z)$, so that we may put $u = f(x,\ n)$;

$$\therefore\ du = \left(\frac{du}{dx}\right)dx + \left(\frac{du}{dn}\right)dn.$$

$$\text{But} \because n = \phi(y, z); \quad \therefore dn = \frac{dn}{dy} dy + \frac{dn}{dz}. dz;$$

$$\therefore du = \frac{du}{dx} dx + \frac{du}{dn} \cdot \frac{dn}{dy} dy + \frac{du}{dn} \cdot \frac{dn}{dz} dz.$$

$$\text{But } \frac{du}{dn} \cdot \frac{dn}{dy} = \frac{du}{dy}, \text{ and } \frac{du}{dn} \cdot \frac{dn}{dz} = \frac{du}{dz};$$

$$\therefore du = \frac{du}{dx} \cdot dx + \frac{du}{dy} \cdot dy + \frac{du}{dz} \cdot dz;$$

and the same method may be extended to any number of variables; whence it appears that the differential of a function of any number of variables equals the sum of the partial differentials.

104. From the preceding reasoning we may find the differentials of a function of two functions of the variable x.

For suppose $u = f(y, z)$, where y and z are functions of x, and therefore u is also a function of x; to find the total differential of u.

Let k and m be the increments of y and z, if x become $x + h$;

$$\therefore u_1 = u + \left(\frac{du}{dy}\right) k + \left(\frac{du}{dz}\right) m + \&c.$$

$$\text{But } k = \frac{dy}{dx} h + \&c.; \quad m = \frac{dz}{dx} h + \&c.;$$

$$u_1 = u + \left\{ \left(\frac{du}{dy}\right) \frac{dy}{dx} + \left(\frac{du}{dz}\right) \cdot \frac{dz}{dx} \right\} h + \&c.;$$

$$\therefore du = \left\{ \left(\frac{du}{dy}\right) \frac{dy}{dx} + \left(\frac{du}{dz}\right) \cdot \frac{dz}{dx} \right\} h;$$

$$\text{or since } \frac{dy}{dx} h = dy, \text{ and } \frac{dz}{dx} h = dz,$$

$$du = \left(\frac{du}{dy}\right) \cdot dy + \left(\frac{du}{dz}\right) dz,$$

which is the total differential of a function of two functions.

105. Again, if $u = f(y, z, v)$, and y, z, v are functions of x; to find the same: let n be the increment of v;

$$\therefore u_1 = u + \left(\frac{du}{dy}\right) k + \left(\frac{du}{dz}\right) m + \left(\frac{du}{dv}\right) n + \&c.$$

$$= u + \left\{ \left(\frac{du}{dy}\right) \frac{dy}{dx} + \left(\frac{du}{dz}\right) \frac{dz}{dx} + \left(\frac{du}{dv}\right) \cdot \frac{dv}{dx} \right\} h + \&c.$$

$$\therefore\ du = \left\{ \left(\frac{du}{dy}\right) \frac{dy}{dx} + \left(\frac{du}{dz}\right) \frac{dz}{dx} + \left(\frac{du}{dv}\right) . \frac{dv}{dx} \right\} h$$

$$= \left(\frac{du}{dy}\right) dy + \left(\frac{du}{dz}\right) dz + \left(\frac{du}{dv}\right) dv.$$

Similarly may the total differential of a function of n functions be found.

Examples.

(1) Let $u = x^m y^n$; find du.

$$\left(\frac{du}{dx}\right) = my^n x^{m-1}; \quad \left(\frac{du}{dy}\right) = nx^m y^{n-1};$$

$$\therefore\ du = \left(\frac{du}{dx}\right) dx + \left(\frac{du}{dy}\right) dy$$

$$= my^n x^{m-1} dx + nx^m y^{n-1} dy$$

$$= x^{m-1} y^{n-1} (mydx + nxdy),$$

$$\frac{d^2u}{dydx} = mny^{n-1}x^{m-1} = \frac{d^2u}{dxdy}.$$

(2) $u = \dfrac{x^m}{y^n}$; $du = \dfrac{x^{m-1}(mydx - nxdy)}{y^{n+1}}$.

(3) $u = \sin(x^2 y)$; find $\dfrac{d^2u}{dxdy}$.

$$\frac{d^2u}{dxdy} = 2x(\cos x^2 y - x^2 y \sin x^2 y). \qquad (?)$$

(4) $u = \sin^{-1}\dfrac{x}{y}$; $du = \dfrac{ydx - xdy}{y\sqrt{y^2 - x^2}}$.

(5) $u = \log\left(\dfrac{x + \sqrt{x^2 - y^2}}{x - \sqrt{x^2 - y^2}}\right)$; $du = 2 . \dfrac{ydx - xdy}{y\sqrt{x^2 - y^2}}$.

(6) $u = x^y$; $du = x^y\left(\dfrac{y}{x} dx + \log x . dy\right)$.

$$\frac{d^2u}{dydx} = x^y\left(\frac{1}{x} + \frac{y}{x}\log x\right) = \frac{d^2u}{dxdy}.$$

(7) $u = \dfrac{x^2 y}{a^2 - z^2}$; find du, and shew that,

$$\frac{d^2u}{dxdy} = \frac{2x}{a^2 - z^2} = \frac{d^2u}{dydx}.$$

$$\frac{d^2u}{dxdz} = \frac{4xyz}{(a^2-z^2)^2} = \frac{d^2u}{dzdx},$$

$$\frac{d^2u}{dydz} = \frac{2x^2z}{(a^2-z^2)^2} = \frac{d^2u}{dzdy},$$

$$\frac{d^3u}{dxdydz} = \frac{4xz}{(a^2-z^2)^2} = \frac{d^3u}{dzdydx} = \frac{d^3u}{dydxdz}.$$

(8) Let $u = \dfrac{1}{\sqrt{(a-x)^2+(b-y)^2+(c-z)^2}}$, shew that

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = 0.$$

Here $\dfrac{1}{u^2} = (a-x)^2 + (b-y)^2 + (c-z)^2$; ....(1)

$$\therefore \frac{du}{dx} = u^3(a-x);$$

$$\therefore \frac{d^2u}{dx^2} = 3u^2\frac{du}{dx}(a-x) - u^3 = 3u^5(a-x)^2 - u^3,$$

$$\therefore \frac{d^2u}{dy^2} = 3u^5(b-y)^2 - u^3,$$

$$\frac{d^2u}{dz^2} = 3u^5(c-z)^2 - u^3;$$

$$\therefore \frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = 3u^5\frac{1}{u^2} - 3u^3 = 0,$$

an equation of great importance in physical science.

106. Let u be a homogenous function of x, y, z, &c. and let n be the sum of the exponents in each term, then

$$nu = \frac{du}{dx}x + \frac{du}{dy}y + \frac{du}{dz}z + \&c.$$

For x, y, z, &c. put $x+mx$, $y+my$, $z+mz$, &c. then u becomes $(1+m)^n u$;

$$\therefore (1+m)^n u = u + \frac{du}{dx}mx + \frac{d^2u}{dx^2}\cdot\frac{m^2x^2}{1\cdot 2} + \&c.$$

$$+\frac{du}{dy}my + \frac{d^2u}{dydx}m^2xy + \&c.$$

$$+\frac{du}{dz}mz + \frac{d^2u}{dzdx}\cdot m^2zx + \&c.$$

$$+\frac{d^2u}{dy^2}\frac{m^2y^2}{1\cdot 2} + \&c.$$

$$+ \&c.,$$

$$\text{also } = u + num + n\frac{n-1}{2}um^2 + \&c.;$$

$$\therefore\ nu = \frac{du}{dx}x + \frac{du}{dy}y + \frac{du}{dz}z + \&c.,$$

$$\text{and } n(n-1)u = \frac{d^2u}{dx^2}x^2 + \frac{d^2u}{dy^2}y^2 + \frac{d^2u}{dz^2}z^2 + \&c.$$

$$+ 2\frac{d^2u}{dydx}xy + 2\frac{d^2u}{dzdx}zx + 2\frac{d^2u}{dzdy}zy + \&c.$$

(1) Let $u = (x+y+z)^3$; here $n = 3$;

$$\therefore\ \frac{du}{dx} = \frac{du}{dy} = \frac{du}{dz} = 3(x+y+z)^2;$$

$$\therefore\ \frac{du}{dx}x + \frac{du}{dy}y + \frac{du}{dz}z = 3(x+y+z)^2(x+y+z) = 3u.$$

(2) Let $u = \dfrac{xyz}{x+y+z}$; here $n = 2$,

$$\text{and } \frac{du}{dx}x + \frac{du}{dy}y + \frac{du}{dz}z = 2u.$$

(3) Let $u = \dfrac{x^3+y^3}{x-y}$; here $n = 2$,

$$\text{and } \frac{du}{dx}x + \frac{du}{dy}y = 2u.$$

(4) Let $u = \dfrac{1}{x^2+y^2}$; here $n = -2$,

$$\text{and } \frac{du}{dx}x + \frac{du}{dy}y = -2u.$$

(5) Let $u = \dfrac{\sqrt{x}+\sqrt{y}}{x+y}$; here $n = -\frac{1}{2}$,

$$\text{and } \frac{du}{dx}x + \frac{du}{dy}y = -\tfrac{1}{2}u.$$

(6) Let $u = \dfrac{xy}{ax+bz}$; here $n = 1$,

$$\text{and } \frac{du}{dx}x + \frac{du}{dy}y + \frac{du}{dz}z = u.$$

(7) Let $u = \sin^{-1}\sqrt{\dfrac{x-y}{x+y}}$; here $n = 0$,

$$\text{and } \frac{du}{dx}x + \frac{du}{dy}y = 0.$$

H

(8) Let $u=(x^2+y^2)^2$; here $n=4$;

$$\therefore \frac{d^2u}{dx^2}x^2+2\frac{d^2u}{dydx}xy+\frac{d^2u}{dy^2}y^2=12u.$$

Implicit Functions.

107. When there is an implicit function of y and x, it is frequently impossible to solve the equation with respect to y, and obtain $y=f(x)$; but by considering $f(x, y,)=0$ to be a function of two variables, we may from the preceding expansions for such functions obtain rules easy of application.

Let $u=f(x, y)=0$, and let u_1 represent u when x becomes $x+h$, and therefore y becomes $y+k$;

$$\therefore\ u_1=u+\left(\frac{du}{dx}\right)h+\left(\frac{du}{dy}\right)k+\&c.$$

But $\because\ u=0$, whatever x and y are; $\therefore\ u_1=0$;

$$\therefore\ 0=\left(\frac{du}{dx}\right)h+\left(\frac{du}{dy}\right)k+\&c.$$

But $k=\frac{dy}{dx}h+\&c.$, $\because\ y=f(x)$;

$$\therefore\ 0=\left\{\left(\frac{du}{dx}\right)+\left(\frac{du}{dy}\right)\frac{dy}{dx}\right\}h+Bh^2+Ch^3+\&c.;$$

$$\therefore\ \left(\frac{du}{dx}\right)+\left(\frac{du}{dy}\right)\frac{dy}{dx}=0,$$

whence $\frac{dy}{dx}$ may be found from $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$.

Ex. $y^3-3axy+x^3=0$; find $\frac{dy}{dx}$.

Let $u=y^3-3axy+x^3$;

$$\therefore\ \left(\frac{du}{dx}\right)=-3ay+3x^2;\quad \left(\frac{du}{dy}\right)=3y^2-3ax.$$

$$\text{But } \left(\frac{du}{dx}\right)+\left(\frac{du}{dy}\right)\cdot\frac{dy}{dx}=0;$$

$$\therefore\ -3ay+3x^2+(3y^2-3ax)\frac{dy}{dx}=0;\quad \therefore\ \frac{dy}{dx}=\frac{ay-x^2}{y^2-ax}.$$

Cor. 1. Since $\left\{\left(\frac{du}{dx}\right)+\left(\frac{du}{dy}\right)\cdot\frac{dy}{dx}\right\}h=0$;

$$\therefore \left(\frac{du}{dx}\right) dx + \left(\frac{du}{dy}\right) \cdot \frac{dy}{dx}\, dx = 0; \text{ putting } dx \text{ for } h;$$

$$\text{or } \left(\frac{du}{dx}\right) dx + \left(\frac{du}{dy}\right) dy = 0. \quad \{\because y = f(x)\}; \quad \therefore du = 0.$$

Cor. 2. Hence, since if $u = 0$, $du = 0$; $\therefore$ if $du = 0$, $d^2u = 0$; and thus if $u = 0$, $d^nu = 0$.

108. From the equation $\left(\frac{du}{dx}\right) + \left(\frac{du}{dy}\right) \cdot \frac{dy}{dx} = 0$ (1), to find $\frac{d^2u}{dx^2}$ (where d^2u means the second total differential of u), and thence to deduce $\frac{d^2y}{dx^2}$.

$$\text{Since } \frac{du}{dx} = \left(\frac{du}{dx}\right) + \left(\frac{du}{dy}\right) \cdot \frac{dy}{dx} = 0;$$

$$\therefore \frac{d^2u}{dx^2} = \frac{d}{dx} \cdot \left(\frac{du}{dx}\right) + \frac{d}{dx} \left\{ \left(\frac{du}{dy}\right) \cdot \frac{dy}{dx} \right\} \text{ .... } (2).$$

Put in (1) $\frac{du}{dx}$ for u, and then $\frac{du}{dy}$ for u;

$$\therefore \frac{d}{dx} \cdot \left(\frac{du}{dx}\right) = \left(\frac{d^2u}{dx^2}\right) + \frac{d^2u}{dxdy} \cdot \frac{dy}{dx},$$

$$\frac{d}{dx} \left\{ \left(\frac{du}{dy}\right) \cdot \frac{dy}{dx} \right\} = \left(\frac{du}{dy}\right) \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{d}{dx} \cdot \left(\frac{du}{dy}\right)$$

$$= \left(\frac{du}{dy}\right) \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \left\{ \frac{d^2u}{dxdy} + \frac{d^2u}{dy^2} \cdot \frac{dy}{dx} \right\};$$

$\therefore$ substituting in (2), we have

$$\frac{d^2u}{dx^2} = \left(\frac{d^2u}{dx^2}\right) + 2\, \frac{d^2u}{dxdy} \cdot \frac{dy}{dx} + \left(\frac{du}{dy}\right) \cdot \frac{d^2y}{dx^2} + \left(\frac{d^2u}{dy^2}\right) \cdot \frac{dy^2}{dx^2} = 0 \ldots (3),$$

and because from equation (1) $\frac{dy}{dx}$ may be found in terms of the partial differential coefficients $\left(\frac{du}{dx}\right)$, $\left(\frac{du}{dy}\right)$, and $\left(\frac{d^2u}{dx^2}\right)$ and $\left(\frac{d^2u}{dy^2}\right)$ and $\frac{d^2u}{dxdy}$ being similarly found, $\frac{d^2y}{dx^2}$ may be determined. In the same manner $\frac{d^3u}{dx^3}$, and $\therefore$ $\frac{d^3y}{dx^3}$ and differential coefficients of higher orders may be found.

109. Next, let $u = 0$ be a function of three variables x, y, z, or let z be an implicit function of (x, y); and let

$z+m$ be the value of z when the independent variables, x and y, become respectively $x+h$ and $y+k$;

$$\therefore \text{ since } u_1 = 0 = f(x+h,\ y+k,\ z+m),$$

$$0 = \left(\frac{du}{dx}\right)h + \left(\frac{du}{dy}\right)k + \left(\frac{du}{dz}\right)m + Ah^2 + Bk^2 + Cm^2 + \&c.$$

But $z+m = \phi(x+h,\ y+k)$;

$$\therefore\ m = \left(\frac{dz}{dx}\right)h + \left(\frac{dz}{dy}\right)k + \&c.;$$

$$\therefore\ 0 = \left\{\left(\frac{du}{dx}\right) + \left(\frac{du}{dz}\right).\frac{dz}{dx}\right\}h + \left\{\left(\frac{du}{dy}\right) + \left(\frac{du}{dz}\right).\frac{dz}{dy}\right\}k + \&c.;$$

$$\therefore\ \left(\frac{du}{dx}\right) + \left(\frac{du}{dz}\right).\frac{dz}{dx} = 0 \ \ldots\ldots\ldots\ldots (1),$$

$$\left(\frac{du}{dy}\right) + \left(\frac{du}{dz}\right).\frac{dz}{dy} = 0 \ \ldots\ldots\ldots\ldots (2);$$

whence $dz = \frac{dz}{dx}dx + \frac{dz}{dy}dy$ may be found.

110. The differential coefficients of the superior orders, can be found by differentiating the equations

$$\left(\frac{du}{dx}\right) + \left(\frac{du}{dz}\right).\frac{dz}{dx} = 0 \ \ldots\ldots\ldots\ldots (1).$$

$$\left(\frac{du}{dy}\right) + \left(\frac{du}{dz}\right).\frac{dz}{dy} = 0 \ \ldots\ldots\ldots\ldots (2).$$

111. Thus to obtain $\frac{d^2z}{dx^2}$, $\frac{d^2z}{dydx}$ and $\frac{dz^2}{dy^2}$.

Consider equations (1) and (2) and $\left(\frac{du}{dx}\right)$, $\left(\frac{du}{dy}\right)$ and $\left(\frac{du}{dz}\right)$ as functions of x, y, z.

(1°) Let equation (1) be differentiated with respect to x; it must be considered as a function of x and z, and therefore from (Art. 108), putting z for y,

$$\left(\frac{d^2u}{dx^2}\right) + 2.\frac{d^2u}{dz.dx}.\frac{dz}{dx} + \left(\frac{d^2u}{dz^2}\right).\frac{dz^2}{dx^2} + \left(\frac{du}{dz}\right).\frac{d^2z}{dx^2} = 0 \ldots (3).$$

(2°) Differentiate (2), considered a function of y and z. Write y for x in equation (3);

$$\left(\frac{d^2u}{dy^2}\right) + 2\frac{d^2u}{dzdy}.\frac{dz}{dy} + \left(\frac{d^2u}{dz^2}\right).\frac{dz^2}{dy^2} + \left(\frac{du}{dz}\right).\frac{d^2z}{dy^2} = 0 \ldots (4).$$

Now either differentiate (1) with respect to y, or (2) with respect to x; and since in the former case $\frac{dz}{dx}$ becomes $\frac{d^2z}{dydx}$, and in the latter $\frac{dz}{dy}$ becomes $\frac{d^2z}{dxdy}$, and that $\frac{d^2z}{dydx}=\frac{d^2z}{dxdy}$, the results will be identical.

Let equation (1) be differentiated with respect to y; and to do this put $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dz}\right)$ for u in equation (2);

$$\therefore \frac{d^2u}{dx\,.\,dy}+\frac{d^2u}{dx\,.\,dz}\cdot\frac{dz}{dy}+\frac{d^2u}{dz\,.\,dy}\cdot\frac{dz}{dx}$$
$$+\left(\frac{d^2u}{dz^2}\right)\cdot\frac{dz}{dx}\cdot\frac{dz}{dy}+\left(\frac{dx}{dz}\right)\cdot\frac{d^2z}{dy\,.\,dx}=0\ldots(5).$$

From (3), (4), (5), $\frac{d^2z}{dx^2}$, $\frac{d^2z}{dy^2}$ and $\frac{d^2z}{dydx}$ may be found, and $\frac{dz}{dx}$ and $\frac{dz}{dy}$ from (1) and (2).

112. From this it is obvious, that if $u=f(x, y, z)$, be an explicit function of x, y and z, where z is a function of x and y, that the differential coefficients of u may be found.

$$\text{For } \frac{du}{dx}=\left(\frac{du}{dx}\right)+\left(\frac{du}{dz}\right)\cdot\frac{dz}{dx}\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\frac{du}{dy}=\left(\frac{du}{dy}\right)+\left(\frac{du}{dz}\right)\cdot\frac{dz}{dy}\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$\frac{d^2u}{dx^2}=\left(\frac{d^2u}{dx^2}\right)+2\frac{d^2u}{dzdx}\cdot\frac{dz}{dx}+\left(\frac{d^2u}{dz^2}\right)\cdot\frac{dz^2}{dx^2}+\left(\frac{du}{dz}\right)\cdot\frac{d^2z}{dx^2}\cdots(3),$$

$$\frac{d^2u}{dy^2}=\left(\frac{d^2u}{dy^2}\right)+2\frac{d^2u}{dzdy}\cdot\frac{dz}{dy}+\left(\frac{d^2u}{dz^2}\right)\cdot\frac{dz^2}{dy^2}+\left(\frac{du}{dz}\right)\cdot\frac{d^2z}{dy^2}\ldots(4),$$

$$\frac{d^2u}{dxdy}=\frac{d^2u}{dxdy}+\frac{d^2u}{dxdz}\cdot\frac{dz}{dy}+\frac{d^2u}{dydz}\cdot\frac{dz}{dx}$$
$$+\left(\frac{d^2u}{dz^2}\right)\cdot\frac{dz}{dx}\cdot\frac{dz}{dy}+\frac{dx}{dz}\cdot\frac{d^2z}{dydx}\ldots\ldots\ldots\ldots\ldots(5).$$

From which $\frac{dz}{dx}$, $\frac{dz}{dy}$, $\frac{d^2z}{dx^2}$, $\frac{d^2z}{dy^2}$, $\frac{d^2z}{dydx}$ may be found.

By a similar process the differentials of the third and higher orders may be obtained.

Elimination by means of Differentiation.

113. We have seen that if a constant quantity be connected with the function by the signs $\pm$, it disappears from the differential coefficients. Should it however be multiplied into the function or any term of the function, it will still appear in the value of the differential coefficient.

Thus if $u=0$ be a function of x and y, involving a constant a, both $u=0$ and $du=0$ will contain a, but between these two equations it may be eliminated, and an equation will arise independent of a, which is called a differential equation.

$$\text{Thus, let } y=ax^2; \quad \therefore \frac{dy}{dx}=2ax=\frac{2y}{x};$$

an equation from which a has disappeared.

Irrational and transcendental quantities may also be eliminated by differentiation.

$$\text{Thus, let } y=(a^2+x^2)^{\frac{m}{n}};$$

$$\therefore \frac{dy}{dx}=2x \cdot \frac{m}{n} \cdot (a^2+x^2)^{\frac{m}{n}-1}=\frac{2mx\,(a^2+x^2)^{\frac{m}{n}}}{n\,(a^2+x^2)}=\frac{2mxy}{n\,(a^2+x^2)}.$$

If there be two constants as a and b involved in the equation $y=f(x)$; then the equations $u=0$, $du=0$, and $d^2u=0$ must be combined, to eliminate them.

Ex. 1. $u=y-ax^2-bx=0$, or $y=ax^2+bx$;

$$\therefore \frac{dy}{dx}=2ax+b; \quad \frac{d^2y}{dx^2}=2a; \quad b=\frac{dy}{dx}-x\frac{d^2y}{dx^2};$$

$$\therefore y-\frac{x^2}{2}\cdot\frac{d^2y}{dx^2}-x\frac{dy}{dx}+x^2\frac{d^2y}{dx^2}=0;$$

$$\therefore \frac{d^2y}{dx^2}-\frac{2}{x}\cdot\frac{dy}{dx}+\frac{2y}{x^2}=0.$$

Ex. 2. $y=a \cdot \cos mx+b \cdot \sin mx$; eliminate a and b.

$$\frac{dy}{dx}=-ma\sin mx+mb\cos mx,$$

$$\frac{d^2y}{dx^2}=-m^2a\cos mx-m^2b\sin mx$$

$$=-m^2\{a\cos mx+b\sin mx\}=-m^2y;$$

$$\therefore \frac{d^2y}{dx^2}+m^2y=0.$$

Ex. 3. $y = ae^{2x}\sin(3x+b)$; eliminate a and b.

$$\frac{dy}{dx} = 2ae^{2x}\sin(3x+b) + 3ae^{2x}\cos(3x+b)$$
$$= 2y + 3y\cot(3x+b),$$
$$\frac{d^2y}{dx^2} = 4ae^{2x}\sin(3x+b) + 6ae^{2x}\cos(3x+b)$$
$$+ 6ae^{2x}\cos(3x+b) - 9ae^{2x}\sin(3x+b)$$
$$= -5y + 12y\cot(2x+b)$$
$$= -5y + 4\frac{dy}{dx} - 8y;$$
$$\therefore \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0.$$

114. If $u = f(xyz) = 0$, or $z = f(xy)$; we may by means of the partial differential coefficients $\frac{dz}{dy}$ and $\frac{dz}{dx}$ eliminate two quantities from $z = f(xy)$, and by proceeding to the second differential, have three other equations for $\frac{d^2z}{dx^2}$, $\frac{d^2z}{dydx}$ and $\frac{d^2z}{dy^2}$, and therefore five quantities may be eliminated.

Ex. 4. Let $z = f(ax+by)$; eliminate the arbitrary function.

$$\text{Let } ax = by = v; \therefore z = f(v),$$
$$\text{and } \frac{dz}{dx} = \frac{dz}{dv}\cdot\frac{dv}{dx}. \quad \text{But } \frac{dz}{dv} = f'(v), \text{ and } \frac{dv}{dx} = a;$$
$$\therefore \frac{dz}{dx} = af'(v); \quad \frac{dz}{dy} = \frac{dz}{dv}\cdot\frac{dv}{dy} = f'(v).b,$$
$$\text{or } b.\frac{dz}{dx} = abf'(v), \quad \text{and } a.\frac{dz}{dy} = ab.f'(v);$$
$$\therefore b\frac{dz}{dx} - a\frac{dz}{dy} = 0, \quad \text{or } bp - aq = 0.$$

As an example. Let $z = \sin(ax+by)$;

$$\therefore p = a\cos(ax+by), \quad q = b\cos(ax+by);$$
$$\therefore bp - aq = 0.$$

Ex. 5. Let $z=(x+y)^m\phi(x^2-y^2)$; eliminate the function.

$$p = m.(x+y)^{m-1}\phi(x^2-y^2) + 2(x+y)^m\phi'(x^2-y^2).x\ldots(1),$$
$$q = m.(x+y)^{m-1}\phi(x^2-y^2) - 2(x+y)^m\phi'(x^2-y^2).y\ldots(2).$$

Multiply (1) by (y), and (2) by x, and add;

$$\therefore\ yp + xq = m(x+y)^m\phi(x^2 - y^2) = mz,$$

$$\text{or } py + qx = mz.$$

Ex. 6. Let $zx = f\left(\frac{x}{y}\right)$; or $z = \frac{1}{x}\cdot f\left(\frac{x}{y}\right)$;

$$\therefore\ p = -\frac{1}{x^2} f\left(\frac{x}{y}\right) + \frac{1}{xy}\cdot f'\left(\frac{x}{y}\right),$$

$$q = \ldots\ldots\ldots\ldots\ldots - \frac{1}{y^2}\cdot f'\left(\frac{x}{y}\right);$$

$$px + z = \frac{1}{y}\cdot f'\left(\frac{x}{y}\right);\quad qy = -\frac{1}{y}\cdot f'\left(\frac{x}{y}\right);$$

$$\therefore\ px + qy + z = 0.$$

Ex. 7. Let $z = f(y+ax) + \phi(y-ax)$,

$$p = a\,.\,f'(y+ax) - a\phi'(y-ax),$$

$$q = f'(y+ax) + \phi'(y-ax);$$

$$\therefore\ \frac{d^2z}{dx^2} = a^2 f''(y+ax) + a^2\phi''(y-ax),$$

$$\frac{d^2z}{dy^2} = f''(y+ax) + \phi''(y-ax);$$

$$\therefore\ \frac{d^2z}{dx^2} - a^2\,.\,\frac{d^2z}{dy^2} = 0.$$

The equation to vibrating chords.

Ex. 8. If $z = xf(a) + y\phi(a) + \psi(a)$ .... (1),

$0 = xf'(a) + y\phi'(a) + \psi'(a)$ .... (2),

where $f'(a) = \frac{d\,.\,f(a)}{da}$, to eliminate the arbitrary functions.

Differentiate (1) successively with respect to x and y, considering a as a function of x and y;

$$\therefore\ \frac{dz}{dx} = f(a) + \{xf'(a) + y\phi'(a) + \psi'(a)\}\frac{da}{dx},$$

$$\frac{dz}{dy} = \phi(a) + \{xf'(a) + y\phi'(a) + \psi'(a)\}\frac{da}{dy};$$

therefore from (2), $\frac{dz}{dx} = f(a)$; $\frac{dz}{dy} = \phi(a)$;

$$\therefore\ \frac{dz}{dx} = F\left(\frac{dz}{dy}\right);$$

$$\therefore\ \frac{d^2z}{dx^2} = F'\left(\frac{dz}{dy}\right) \cdot \frac{d^2z}{dxdy} \text{ .... } (3),$$

$$\frac{d^2z}{dydx} = F'\left(\frac{dz}{dy}\right) \cdot \frac{d^2z}{dy^2} \text{ .... } (4),$$

whence multiplying crossways,

$$\left(\frac{d^2z}{dx^2}\right)\left(\frac{d^2z}{dy^2}\right) = \left(\frac{d^2z}{dxdy}\right)^2,$$

the equation to developable surfaces.

Ex. 9. $y = xe^{cx}$; eliminate c.

$$xdy - ydx = y(\log y - \log x)dx.$$

Ex. 10. Eliminate a and b from $y^2 = ax + bx^2$.

Ex. 11. If $y = a\sin x + b\sin 2x$, shew that

$$\frac{d^4y}{dx^4} + 5\frac{d^2y}{dx^2} + 4y = 0.$$

Ex. 12. If $z = \frac{x^2}{2} + f(y + \log x)$; $px - q = x^2$.

Ex. 13. If $z = f\left(\frac{y^2 - x^2}{x}\right)$; $2xyp + (x^2 + y^2)q = 0$.

Ex. 14. $x^2 + y^2 + z^2 = f(ax + by + z)$.

$$(y - bz)p - (x - az)q = bx - ay.$$

Ex. 15. $z = ax + by + c$; eliminate a, b, c.

$$\frac{d^3z}{dx^3} \cdot \frac{d^2y}{dx^2} - \frac{d^2z}{dx^2} \cdot \frac{d^3y}{dx^3} = 0.$$

Ex. 16. If $z = xf\left(\frac{y}{x}\right) + \phi(xy)$, shew that

$$x^2 \cdot \frac{d^2z}{dx^2} = y^2 \cdot \frac{d^2z}{dy^2}.$$

CHAPTER IX.

Maxima and Minima of Functions of two Variables.

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115. If $u=f(x, y)$ be an equation between the function u, and the two independent variables, x and y, there may be some particular value of x, and also of y, which will make the function greater or less than the values which immediately precede or follow it. It is then a maximum or minimum. We proceed to find the relation between the differential coefficients, when this circumstance takes place.

116. Let u_1 be the value of u, when $x+h$ and $y+k$ are written for x and y respectively; and u_2 the value of u, when $x-h$ and $y-k$ are substituted for the same quantities. Also put A for $\frac{d^2u}{dx^2}$, B for $\frac{d^2u}{dydx}$, and C for $\frac{d^2u}{dy^2}$. Then

$$u_1 = u + \frac{du}{dx}h + \frac{du}{dy}k + \tfrac{1}{2}\{Ah^2 + 2Bhk + Ck^2\} + \&c.$$

$$\text{and } u_2 = u - \left(\frac{du}{dx}h + \frac{du}{dy}k\right) + \tfrac{1}{2}\{Ah^2 + 2Bhk + Ck^2\} - \&c.$$

Now since the values of h and k may be assumed so small that, (as long as the differential coefficients $\frac{du}{dx}$ and $\frac{du}{dy}$ remain finite) the algebraical sign of $u_1 - u$ and $u_2 - u$ will depend upon that of the term $\left(\frac{du}{dx}h + \frac{du}{dy}k\right)$, it is manifest, that if this term exist, $u_1 - u$ and $u_2 - u$ cannot be both positive or both negative, or there cannot be a minimum or maximum of u. Therefore at a maximum or minimum $\frac{du}{dx}h + \frac{du}{dy}k$ must $= 0$. A condition which can only be fulfilled, since h and k are independent quantities, by making $\frac{du}{dx} = 0$, and $\frac{du}{dy} = 0$. Hence at a maximum or minimum,

$$\begin{aligned} u_1 - u &= \tfrac{1}{2}(Ah^2 + 2Bhk + Ck^2) + \&c. \\ &= \frac{h^2}{1.2}\{A + 2Bn + Cn^2\} + \&c., \text{ if } k = nh. \end{aligned}$$

Therefore the sign of $u_1 - u$, and also of $u_2 - u$, will depend upon that of the coefficient of $\frac{h^2}{2}$, that is, upon $A + 2Bn + Cn^2$. Hence, this term must not change its sign whatever be the value of n; which it will not do, if it can be put under the form of the sum of two squares, as $(n+a)^2 + \beta^2$.

$$\text{Now } A + 2Bn + Cn^2 = \frac{1}{C}\{CA + 2BCn + C^2n^2\}$$
$$= \frac{1}{C}\{CA - B^2 + (B + Cn)^2\}$$
$$= \frac{1}{C}\left\{CA - B^2 + C^2\left(\frac{B}{C} + n\right)^2\right\},$$

which is of the requisite form, if CA be not less than B^2: or to have a maximum or minimum of a function of two variables, we must first have $\frac{du}{dx} = 0$ and $\frac{du}{dy} = 0$; and secondly, $\frac{d^2u}{dx^2} \times \frac{d^2u}{dy^2}$ not less than $\left(\frac{d^2u}{dydx}\right)^2$.

117. It is obvious that $\frac{d^2u}{dx^2}$ and $\frac{d^2u}{dy^2}$ must have the same algebraical sign; and also if they be both *negative*, u is a maximum, if both *positive*, u is a minimum.

If the second differential coefficient of $u = 0$, when the first does, there will not be a maximum or minimum, unless the third differential coefficient vanishes, and the fourth neither vanishes nor changes its sign, whatever be the value of n.

Ex. 1. Let $u = x^3 + y^3 - 3axy$,

$$\frac{du}{dx} = 3x^2 - 3ay = 0; \quad \therefore\ y = \frac{x^2}{a},$$

$$\frac{du}{dy} = 3y^2 - 3ax = 0; \quad \therefore\ y^2 - xa = \frac{x^4}{a^2} - ax = 0;$$

$\therefore\ x = 0$, and $x^3 - a^3 = 0$; whence $x = a$; the other two roots are impossible; and $y = \frac{x^2}{a} = a$; or $= 0$.

$$\text{Also } \frac{d^2u}{dx^2} = 6x, \quad \frac{d^2u}{dy^2} = 6y, \text{ and } \frac{d^2u}{dydx} = -3a.$$

If $x = 0$, $A = 0$, $C = 0$, and $B = -3a$.

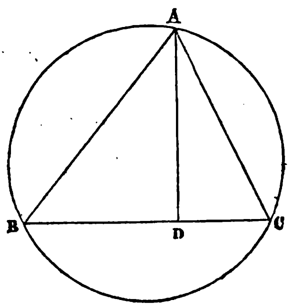
$$\text{If } x = a, \quad A = 6a, \quad C = 6a, \qquad B = -3a,$$
$$AC = 36a^2, \text{ and } B^2 = 9a^2;$$

$\therefore x = a$, $\because A$ is $+$, gives a minimum, and $u = -a^3$; $x = 0$ gives neither a maximum nor minimum.

Ex. 2. $u = x^3y^2(a - x - y)$.

$$\frac{du}{dx} = 3x^2y^2(a - x - y) - x^3y^2 = 0,$$
$$\frac{du}{dy} = 2x^3y(a - x - y) - x^3y^2 = 0;$$
$$\therefore 3(a - x - y) = x; \quad 2(a - y - x) = y; \quad \therefore 2x = 3y;$$
$$\therefore 3a - 3x - 2x = x, \quad \text{or } x = \frac{a}{2},$$
$$2a - 3y - 2y = y, \quad \text{or } y = \frac{a}{3};$$
$$\therefore a - x - y = a - \frac{a}{2} - \frac{a}{3} = \frac{a}{6},$$
$$\frac{d^2u}{dx^2} = 6xy^2(a - x - y) - 6x^2y^2 = 6\left\{\frac{a}{2}\cdot\frac{a^2}{9}\cdot\frac{a}{6} - \frac{a^2}{4}\cdot\frac{a^2}{9}\right\} = -\frac{a^4}{9},$$
$$\frac{d^2u}{dy^2} = 2x^3(a - x - y) - 4x^3y = 2\left\{\frac{a^3}{8}\cdot\frac{a}{6} - 2\cdot\frac{a^3}{8}\cdot\frac{a}{3}\right\} = -\frac{a^4}{8},$$
$$\frac{d^2u}{dxdy} = 6x^2y(a - x - y) - 3x^2y^2 - 2x^3y = \frac{a^4}{12} - \frac{a^4}{12} - \frac{a^4}{12} = -\frac{a^4}{12};$$
$$\therefore AC = \frac{a^8}{72}, \text{ and } B^2 = \frac{a^8}{144}; \quad \therefore AC \text{ is } > B^2;$$
$$\text{and } u = \frac{a^3}{8} \times \frac{a^2}{9} \times \frac{a}{6} = \frac{a^6}{432} \text{ is a maximum, } \because A \text{ is } -.$$

Ex. 3. Inscribe the greatest triangle within a given circle.



R the radius,

a, b, c the sides,

$\theta = \angle B$, $\phi = \angle C$.

$$\text{But } u = \frac{bc}{2}\cdot\sin A$$
$$= \frac{bc}{2}\cdot\sin(\theta + \phi),$$
$$\text{and } b = 2R\cdot\sin\theta;$$
$$c = 2R\cdot\sin\phi;$$

$$\therefore\ u = 2R^2 \sin\theta \sin\phi \sin(\phi+\theta) = \text{maximum};$$

$$\therefore\ \frac{du}{d\theta} = 2R^2\{\cos\theta . \sin(\phi+\theta) + \sin\theta . \cos(\phi+\theta)\} \sin\phi = 0,$$

$$\text{and } \frac{du}{d\phi} = 2R^2\{\cos\phi . \sin(\phi+\theta) + \sin\phi . \cos(\phi+\theta)\} \sin\theta = 0;$$

$$\therefore\ \sin(\phi+2\theta) = 0 = \sin\pi, \text{ and } \sin(\theta+2\phi) = 0 = \sin\pi;$$

$$\therefore\ \phi + 2\theta = \pi, \text{ and } \theta + 2\phi = \pi;$$

$$\therefore\ \theta - \phi = 0, \text{ or } \theta = \phi;\ \therefore\ 3\theta = \pi, \text{ and } \theta = 60^\circ = \phi,$$

and $\therefore A = 60^\circ$; and the triangle is equiangular.

Ex. 4. Inscribe the greatest parallelopipedon within a given ellipsoid.

Let $2x$, $2y$, $2z$ be the edges,

$2a$, $2b$, $2c$ the principal diameters of the ellipsoid;

$$\therefore\ u = 8xyz, \text{ and } \frac{z^2}{c^2} + \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

$$\therefore\ z^2 = c^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right);$$

$$\therefore\ \frac{du}{dx} = 8yz + 8yx\frac{dz}{dx} = 0,$$

$$\frac{du}{dy} = 8xz + 8yx\frac{dz}{dy} = 0.$$

$$\text{But } \frac{dz}{dx} = -\frac{x}{z}.\frac{c^2}{a^2}, \text{ and } \frac{dz}{dy} = -\frac{y}{z}.\frac{c^2}{b^2};$$

$$\therefore\ z - \frac{x^2}{z}.\frac{c^2}{a^2} = 0, \text{ and } z - \frac{y^2}{z}.\frac{c^2}{b^2} = 0;$$

$$\therefore\ \frac{z^2}{c^2} = \frac{x^2}{a^2}, \text{ and } \frac{z^2}{c^2} = \frac{y^2}{b^2} = \therefore\ \frac{x^2}{a^2};$$

$$\frac{3x^2}{a^2} = 1,\ x = \frac{a}{\sqrt{3}};\ \therefore\ \frac{y^2}{b^2} = \frac{z^2}{c^2} = \tfrac{1}{3};\ \therefore\ y = \frac{b}{\sqrt{3}};\ \therefore\ z = \frac{c}{\sqrt{3}};$$

$$\text{and } u = \frac{8}{3}\frac{abc}{\sqrt{3}}.$$

But if V = volume of ellipsoid; $V = \frac{4}{3}\pi abc$;

$$\therefore\ u = \frac{2V}{\pi\sqrt{3}} \text{ or } u : V :: 2 : \pi\sqrt{3}.$$

Ex. 5. If $u = x^4 + y^4 - 4axy^2$; find x and y, when $u =$ maximum or minimum.

$$x = \pm a\sqrt{2}; \quad y = \pm a\sqrt{8},$$

and $x = 0$; $y = 0$; give minima.

Ex. 6. $u = a\{\sin x + \sin y + \sin(x + y)\}$.

$$x = 60; \quad y = 60; \text{ give } u = 3a\frac{\sqrt{3}}{2} \text{ a maximum.}$$

Ex. 7. Given the perimeter of a triangle, shew that its area is greatest when it is equilateral.

Ex. 8. Divide a quantity a into three such parts x, y, z, that $u = x^m y^n z^p$ may be a maximum; and shew that it is a maximum and not a minimum.

$$x = \frac{ma}{m+n+p}; \quad y = \frac{na}{m+n+p}; \quad z = \frac{pa}{m+n+p}.$$

Ex. 9. Given the surface of a rectangular parallelopipedon; find when its content is a maximum.

If x, y, z be the edges of the solid:

Surface $= 2xy + 2xz + 2yz = 6a^2$,

and $u = xyz$ a maximum,

whence $x = y = z = a$; and solid is a cube.

Ex. 10. If the content of the rectangular parallelopipedon be given, find its form when the surface is a minimum.

It is a cube, as in the preceding question.

Ex. 11. Let $u = ax + by + cz$ a maximum,

and $x^2 + y^2 + z^2 = 1$; find x, y, z;

$$\therefore\ u = ax + by + c\sqrt{1 - x^2 - y^2};$$

$$\therefore\ \frac{du}{dx} = a - \frac{cx}{z} = 0; \quad \therefore\ az = cx,$$

$$\frac{du}{dy} = b - \frac{cy}{z} = 0; \quad \therefore\ bz = cy;$$

$$\therefore\ (a^2 + b^2 + c^2)z^2 = c^2(x^2 + y^2 + z^2) = c^2;$$

$$\therefore\ z = \frac{c}{\sqrt{a^2 + b^2 + c^2}}; \quad x = \frac{a}{\sqrt{a^2 + b^2 + c^2}}; \quad y = \frac{b}{\sqrt{a^2 + b^2 + c^2}}.$$

Ex. 12. $u=(x+1)(y+1)(z+1)$;

a maximum where $A=a^x b^y c^z$

$$x=\frac{\log Abc-2\log a}{3\log a};\quad u=\frac{(\log Abca)^3}{27\log a.\log b.\log c}.$$

Ex. 13. Given the radius of a circle described about a triangle; find its form when the perimeter is a maximum.

If θ and ϕ be two of the angles, and if r be the radius of the circle,

$$u=2r\{\sin\theta+\sin\phi+\sin(\theta+\phi)\};$$

whence $\theta=\phi=60$; and the triangle is equilateral.

Ex. 14. Given the sum of the three axes of an ellipsoid; find them when the volume of the ellipsoid is greatest.

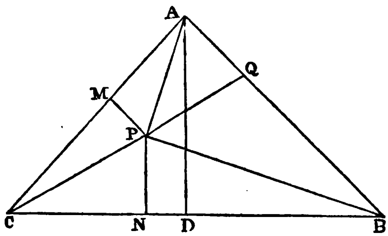
If $2x$, $2y$, $2z$ be the three axes,

$$u=\frac{4}{3}\pi xyz, \text{ a maximum,}$$

and $2x+2y+2z=6a$, the given length; whence

$x=y=z=a$, and ellipsoid becomes a sphere.

Ex. 15. Find that point within a triangle, from which if lines be drawn to the angular points, the sum of their squares shall be a minimum.



Let ABC be a triangle, and P a point within it, a, b, c, the sides of the triangle.

Draw PN, AD perpendicular to CB; join AP, BP, CP.

Let $CN=x$; $NP=y$; then $AD=b\sin C$; $CD=b\cos C$.

Then $CP^2=x^2+y^2$; $BP^2=y^2+(a-x)^2=y^2+x^2+a^2-2ax$,

$$AP^2=(b\cos C-x)^2+(b\sin C-y)^2$$
$$=b^2+x^2+y^2-2b(x\cos C+y\sin C);$$
$$\therefore\ u=3x^2+3y^2+a^2+b^2-2ax-2b(x\cos C+y\sin C);$$
$$\therefore\ x=\tfrac{1}{3}(a+b\cos C);\quad y=\tfrac{1}{3}b\sin C;$$
$$\therefore\ CP=\sqrt{x^2+y^2}=\tfrac{1}{3}\sqrt{2a^2+2b^2-c^2}.$$

The point P is the centre of gravity of the triangle.

Ex. 16. Find a point within a triangle, from which if perpendiculars be let fall upon the sides, the sum of their squares shall be a minimum.

ABC the triangle as before, P the point within it, draw PN, PM, PQ respectively perpendicular to CB, CA, AB.

$$\text{Let } CN = x;\quad PM = p;$$
$$NP = y;\quad PQ = q;$$
$$\text{then } u = y^2 + p^2 + q^2.$$

Now if δ be the perpendicular from a point (α, β) on a line $y = mx + b$,

$$\delta = \frac{\beta - b - m\alpha}{\sqrt{m^2 + 1}}.$$

(1°) If $\delta = p$; $\beta = y$; $\alpha = x$; $b = 0$; $m = \tan C$;

$$\therefore\ p = \frac{y - x\tan C}{\sec C} = y\cos C - x\sin C.$$

(2°) If $\delta = q$; $m = -\tan B$; $b = a\tan B$;

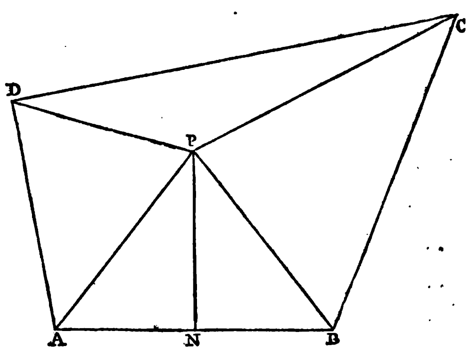
$$\therefore\ q = \frac{y - (a - x)\tan B}{\sec B} = y\cos B - (a - x)\sin B;$$

$$\therefore\ u = y^2 + (y\cos C - x\sin C)^2 + \{y\cos B - (a - x)\sin B\}^2,$$

$$\text{whence } y = \frac{a\sin A \,.\, \sin B \,.\, \sin C}{2(1 - \cos^2 B . \cos^2 C + \sin B \,.\, \sin C \,.\, \cos B \,.\, \cos C)}$$
$$= \frac{abc\sin A}{a^2 + b^2 + c^2};$$

$$\text{and } \therefore\ p = \frac{abc\sin B}{a^2 + b^2 + c^2};\quad q = \frac{abc\sin C}{a^2 + b^2 + c^2}.$$

Ex. 17. Find a point P within a quadrilateral figure $ABCD$, from which if lines be drawn to the angular points, the sum of their squares shall be the least possible.



$$AB = a;\ BC = b;\ AD = c;\ AN = x;\ NP = y;$$

$$\therefore\ u = x^2 + y^2 + y^2 + (a - x)^2 + (b \sin B - y)^2 + (a - x - b \cos B)^2$$
$$+ (c \sin A - y)^2 + (x - c \cos A)^2$$
$$= 4y^2 + 2x^2 + 2(a - x)^2 - 2b\{y \sin B + (a - x) \cos B\}$$
$$+ b^2 + c^2 - 2c(y \sin A + x \cos A);$$

$$\text{whence } x = \tfrac{1}{4}(2a + c \cos A - b \cos B),$$
$$y = \tfrac{1}{4}(c \sin A + b \sin B).$$

Also $\because\ \dfrac{d^2u}{dx^2} = 8 = \dfrac{d^2u}{dy^2};\ \dfrac{d^2u}{dxdy} = 0;$ u is a minimum.

Ex. 18. Of all triangular pyramids of a given base and altitude, to find that which has the least surface.

Let a, b, c be the sides of the base, h the altitude of the pyramid, θ, ϕ, ψ the inclination of the faces to the base.

Then because if p be perpendicular from vertex on the side a, $p \sin \theta = h$, and area of face $= \frac{1}{2} ap = \frac{1}{2} ah \operatorname{cosec} \theta$;

$$\therefore\ u = \tfrac{1}{2} h (a \operatorname{cosec} \theta + b \operatorname{cosec} \phi + c \operatorname{cosec} \psi) \ldots\ldots (1).$$

Also since the base of the pyramid may be divided into three triangles, whose altitudes are respectively

$h \cot \theta$, $h \cot \phi$, $h \cot \psi$; if m^2 be its area,

$$m^2 = \tfrac{1}{2} h (a \cot \theta + b \cot \phi + c \cot \psi) \ldots\ldots\ldots (2),$$

from which combined with (1) a minimum, we have $\theta = \phi = \psi$, or the faces are equally inclined to the base.

Ex. 19. Two points P and Q are given above a plane; find a point R in a plane, such that $PR + RQ$ may be a minimum.

Let the given plane be that of xy; from P and Q draw lines perpendicular to it, let the axis of z pass through P, and the axis of x pass through the foot of the perpendicular from Q.

Then if c = the co-ordinate of P, a and b that of Q, x and y of R;

$$\therefore\ u = PR + QR = \sqrt{x^2 + y^2 + c^2} + \sqrt{y^2 + (a - x)^2 + b^2};$$

$$\therefore\ \frac{du}{dx} = \frac{x}{\sqrt{x^2 + y^2 + c^2}} - \frac{a - x}{\sqrt{y^2 + (a - x)^2 + b^2}} = 0 \ldots (1);$$

$$\therefore\ \frac{du}{dy} = \frac{y}{\sqrt{x^2 + y^2 + c^2}} - \frac{y}{\sqrt{y^2 + (a - x)^2 + b^2}} = 0 \ldots (2).$$

From (2), $y=0$, and therefore the point R is in the axis of x.

From (1), $\frac{x}{PR}=\frac{a-x}{QR}$; or the cosines of the angles which PR and QR make with the axis of x are equal,

$$\text{also } \because \frac{x}{\sqrt{x^2+c^2}}=\frac{a-x}{\sqrt{(a-x)^2+b^2}}; \quad x=\frac{ac}{b+c}.$$

118. When $u=f(xyz)$ is a maximum or minimum, we must put $\frac{du}{dx}=0$; $\frac{du}{dy}=0$; $\frac{du}{dz}=0$; and the equation of condition is $(AC-B^2)(AD-E^2)>(AF-BE)^2$*,

$$\text{where } A=\frac{d^2u}{dx^2}; \quad B=\frac{d^2u}{dydx}; \quad C=\frac{d^2u}{dy^2},$$

$$\text{and } D=\frac{d^2u}{dz^2}; \quad E=\frac{d^2u}{dzdx}; \quad F=\frac{d^2u}{dzdy}.$$

Ex. 20. $u=ax^2y^3z^4-x^3y^3z^4-x^2y^4z^4-x^2y^3z^5=$ a maximum.

$$x=\frac{1}{5}a; \quad y=\frac{3a}{10}; \quad z=\frac{2a}{5}.$$

Ex. 21. $u=\frac{xyz}{(x+a)(x+y)(y+z)(z+e)}=$ maximum.

$$x=\sqrt[4]{a^3e}; \quad y=\sqrt[4]{a^2e^2}; \quad z=\sqrt[4]{ae^3}.$$

* Lacroix, *Calcul. Diff.* Vol. I. Art. 166.

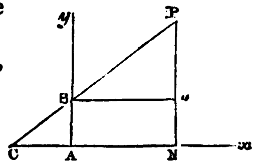
CHAPTER X.

Equations to Curves.

119. We proceed to treat briefly of the equations to a straight line, to the circle, the conic sections, and some other curves, which will be frequently referred to in the succeeding pages; but for complete investigations of the properties of the conic sections and curves in general, we must refer to works expressly written on these subjects: since the object of this Chapter is merely to furnish the student with such a knowledge of the nature of certain curves, as may make the applications of the Differential Calculus to them obvious and interesting.

The Straight Line.

120. Find the equation to the straight line.



Ax, Ay the two axes of x and y,

$\left.\begin{array}{l} AN = x \\ NP = y \\ \angle\, PCA = \theta \end{array}\right\},$ $\quad Bn \perp$ to PN. $\quad AB = b$.

Then $\dfrac{Pn}{Bn} = \dfrac{BA}{CA} = \tan\theta$,

or $\dfrac{y-b}{x} = \tan\theta = m$, by writing m for $\tan\theta$;

$$\therefore\ y = mx + b.$$

Cor. 1. If the line be drawn through a given point; let α and β be the co-ordinates of the point; $\therefore$ if $x = \alpha$, $y = \beta$;

$$\therefore\ \beta = m\alpha + b, \text{ and } y = mx + b;$$

$$\therefore\ y - \beta = m(x - \alpha).$$

Cor. 2. If the line be drawn through the origin, $b = 0$; $\therefore\ y = mx$ is the equation to a line drawn through A.

121. If two lines intersect, find the point of intersection.

Let $y=mx+b$, and $y=m_1x+b_1$ be the equations of the two lines; then, at the point of intersection, the values of the co-ordinates are the same for both lines;

$$\therefore\ mx+b=m_1x+b_1;\quad x=\frac{b_1-b}{m-m_1},$$

$$\text{and } y=\frac{mb_1-mb}{m-m_1}+b=\frac{mb_1-m_1b}{m-m_1}.$$

122. Find the equation to a line passing through two given points.

Let $y=mx+b$ be the equation to the line where m and b are to be determined.

α and β, α_1 and β_1 the co-ordinates of the two points;

$$\therefore\ \beta=m\alpha+b;\quad \beta_1=m\alpha_1+b;$$

$$\therefore\ \beta-\beta_1=m.(\alpha-\alpha_1);\quad \therefore\ m=\frac{\beta-\beta_1}{\alpha-\alpha_1}.$$

But $\because\ y=mx+b$, and $\beta=m\alpha+b$;

$$\therefore\ y-\beta=m.(x-\alpha)=\frac{\beta-\beta_1}{\alpha-\alpha_1}.(x-\alpha).$$

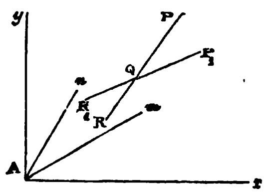
123. To find the angle which two straight lines make with each other at the point of intersection.

$$y=mx+b,\ \text{and}\ y=m_1x+b_1,$$

the equations to the two lines.

PQR and P_1QR_1 the lines.

From A draw An parallel to PR, and Am parallel to $P'R'$;



$\therefore\ \angle\ nAm=\angle\ PQP_1$;

$\therefore\ PQP'=nAx-mAx$

$=\tan^{-1}m-\tan^{-1}m_1$,

$$\text{and }\tan PQP'=\frac{m-m}{1+mm_1}.$$

Cor. 1. If the lines be parallel, $PQP'=0$; $m-m_1=0$; and $m_1=m$;

$$\left.\begin{array}{l}\therefore\ y=mx+b\\ \text{and } y=mx+b_1\end{array}\right\},$$ are the equations to two parallel lines.

Cor. 2. If the lines be perpendicular,

$$\tan PQP'=\frac{1}{0}=\frac{m-m_1}{1+mm_1};$$

$$\therefore\ 1 + mm_1 = 0,\ \text{ and } m_1 = -\frac{1}{m};$$

therefore, if $y = mx + b$ be the equation to a line,

$$y = -\frac{1}{m}x + b_1$$

is the equation to a line perpendicular to it.

124. Find the equation to a line drawn through a given point perpendicular to a given line.

$y = mx + b$, the equation to the given line, α and β the co-ordinates of the given point;

$$\therefore\ y = -\frac{1}{m}x + b_1 \text{ is the equation to the perpendicular,}$$

$$\text{also } \beta = -\frac{1}{m}\alpha + b_1, \text{ since it passes through } (\alpha, \beta);$$

$$\therefore\ (y - \beta) = -\frac{1}{m}(x - \alpha) \text{ is the equation required.}$$

125. Find the perpendicular distance of a given point from a given line.

$y = mx + b$ the equation to the given line, and (β, α) the given point;

$$\therefore\ (y - \beta) = -\frac{1}{m}(x - \alpha)$$

is the equation to a perpendicular from a given point upon the given line.

Then if δ be the distance required, and y_1 and x_1 the co-ordinates of the point of intersection of the given line with the perpendicular,

$$\delta = \sqrt{(x_1 - \alpha)^2 + (y_1 - \beta)^2} = (x_1 - \alpha) \cdot \sqrt{\frac{m^2 + 1}{m}}.$$

$$\text{But } mx_1 + b = \beta - \frac{x_1 - \alpha}{m};$$

$$\therefore\ x_1(m^2 + 1) = m\beta + \alpha - mb;\quad \therefore\ x_1 = \frac{m\beta + \alpha - mb}{m^2 + 1},$$

$$x_1 - \alpha = \frac{m\beta - mb - m^2\alpha}{m^2 + 1} = \frac{m}{m^2 + 1}(\beta - b - m\alpha);$$

$$\therefore\ \delta = \pm\frac{\beta - b - m\alpha}{\sqrt{m^2 + 1}};$$

the upper or lower sign being taken according as the numerator is positive or negative.

Cor. 1. If the point be the origin, then $\alpha = 0$; $\beta = 0$;

$$\therefore \ \delta = \frac{b}{\sqrt{m^2+1}} = \frac{y - mx}{\sqrt{m^2+1}}$$
$$= y \cos\theta - x \sin\theta,$$

since $m = \tan\theta$, and $\sqrt{m^2+1} = \sec\theta$.

Cor. 2. If the line pass through the origin, $b = 0$;

$$\therefore \ \delta = \frac{\beta - m\alpha}{\sqrt{m^2+1}} = \beta\cos\theta - \alpha\sin\theta.$$

Cor. 3. If neither the point be in the origin, nor the line pass through it,

$$\delta = \frac{\beta - m\alpha - y + mx}{\sqrt{m^2+1}}$$
$$= \frac{(\beta - y) - m(\alpha - x)}{\sqrt{m^2+1}}$$
$$= (\beta - y)\cos\theta - (\alpha - x)\sin\theta.$$

126. Find the equation to a straight line, which cuts the axis of y at a distance b from the origin, and the axis of x at a distance a from the origin, in terms of b and a.

$y = mx + b$ the equation to the line,

when $x = 0$, $y = b$,

and $y = 0$, $x = a$; $\therefore \ ma + b = 0$; $\therefore \ m = -\frac{b}{a}$;

$\therefore \ y = -\frac{b}{a}x + b$; or $\frac{y}{b} + \frac{x}{a} = 1$, is the equation.

The Circle.

127. The circle is a curve of which the property is, that every point in its circumference is equidistant from the centre.

Let α and β be the co-ordinates of the centre,

x and y of a point in the curve; a = radius.

Then the distance between two points α, β, and x, y

$$= \sqrt{(x-\alpha)^2 + (y-\beta)^2} = a;$$

$$\therefore \ y^2 + x^2 - 2\beta y - 2\alpha x + \alpha^2 + \beta^2 - a^2 = 0,$$

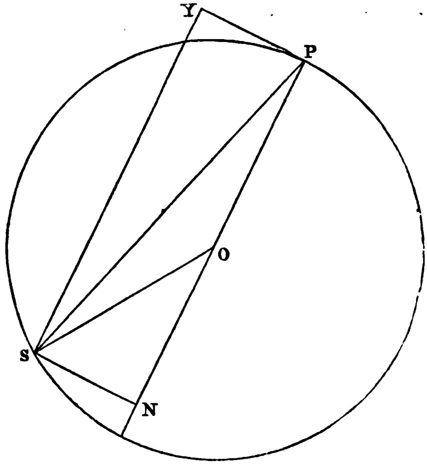
is the equation to the circle.

Cor. 1. If the origin be in the circumference, and the axis of x pass through the centre,

$\beta = 0$, $\alpha = a$; $\therefore \ y^2 + x^2 - 2ax = 0$, or $y^2 = 2ax - x^2$.

COR. 2. If the origin be in the centre,

$\alpha=0, \quad \beta=0; \quad \therefore\ y^2+x^2-a^2=0,$ and $y^2=a^2-x^2$.



128. If S be a point in the circumference of a circle, to find the equation between SP and a perpendicular on the tangent, SY.

Join OP and draw $SN \perp PO$ produced,

$$SP=r; \quad SY=p; \quad OP=a; \quad \text{then } SY=PN.$$

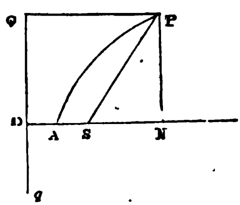
Now $SP^2=SO^2+OP^2+2OP.ON$;

$$\therefore\ r^2=2a^2+2a(p-a)=2ap;$$

$$\therefore\ p=\frac{r^2}{2a} \text{ the equation required.}$$

The Parabola.

129. If from a fixed line QDq perpendicular lines, as QP, be drawn, intersecting lines equal in length, but drawn from a fixed point S, the locus P is the *parabola*.



Draw $SD \perp Qq$, bisect SD in A, then the curve passes through A.

Let $SA=AD=a$,

$AN=x, \quad NP=y$.

Now QP or $DN = SP$; $\therefore DA + AN = \sqrt{NP^2 + SN^2}$;

$$\therefore a + x = \sqrt{y^2 + (x-a)^2};$$

$$\therefore (a+x)^2 \text{ or } (x-a)^2 + 4ax = y^2 + (x-a)^2;$$

$$\therefore y^2 = 4ax.$$

130. The polar equation. Let $SP = r$, $\angle ASP = \theta$.

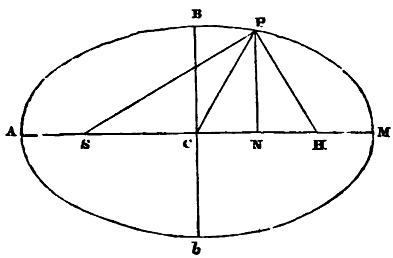
Then $r = DN = 2a + SN = 2a + r\cos PSN$

$$= 2a - r\cos\theta;$$

$$\therefore r = \frac{2a}{1+\cos\theta} = \frac{a}{\cos^2\frac{\theta}{2}}.$$

The Ellipse.

131. If from two fixed points S and H two lines SP



and PH be drawn and intersect, and $SP + PH =$ a constant line, the locus of P is the *ellipse*.

Let $SP + PH = 2a$. Bisect SH in C, and take $CA = CM = a$, the curve passes through A and M. Through C draw $BCb \perp$ to SH. With centre S and radius $= a$ cut this line in the points B and b, the curve will pass through B and b, since HB and Hb each $= a$; make $CB = b$, and let $CS : CA :: e : 1$; $\therefore CS = ae$, which is called the eccentricity.

Let $CN = x$; $NP = y$; $SP = D$; $HP = D_1$;

$$\therefore D^2 = SN^2 + NP^2 = (ae + x)^2 + y^2,$$

$$D_1^2 = HN^2 + NP^2 = (ae - x)^2 + y^2;$$

$$\therefore D^2 + D_1^2 = 2(a^2e^2 + x^2 + y^2), \text{ and } D^2 - D_1^2 = 4aex.$$

But $D + D_1 = 2a$; $\therefore D - D_1 = 2ex$;

$$\therefore\ D = a + ex, \text{ and } D_1 = a - ex;$$

$$\therefore\ D^2 + D_1^2 = 2a^2 + 2e^2x^2 = 2(a^2e^2 + x^2 + y^2);$$

$$\therefore\ y^2 = a^2.(1 - e^2) - x^2.(1 - e^2) = (1 - e^2)(a^2 - x^2).$$

$$\text{But } 1 - e^2 = 1 - \frac{CS^2}{a^2} = \frac{a^2 - CS^2}{a^2} = \frac{SB^2 - CS^2}{a^2} = \frac{b^2}{a^2};$$

$$\therefore\ y^2 = \frac{b^2}{a^2}(a^2 - x^2); \text{ and } \frac{y^2}{b^2} + \frac{x^2}{a^2} = 1.$$

COR. 1. If A be the origin. Make $AN = x_1$;

$$\therefore\ x_1 = a + x, \text{ or } x = x_1 - a;$$

$$\therefore\ a^2 - x^2 = 2ax_1 - x_1^2;$$

$$\therefore\ y^2 = \frac{b^2}{a^2}(2ax_1 - x_1^2).$$

132. If S be the pole, and $ASP = \theta$, and $SP = r$;

$$\therefore\ (2a - r)^2 = HP^2 = HN^2 + NP^2 = (2ae - SN)^2 + r^2 \sin^2\theta,$$

$$\text{and } SN = r\cos PSH = -r\cos\theta;$$

$$\therefore\ 4a^2 - 4ar + r^2 = (2ae + r\cos\theta)^2 + r^2\sin^2\theta$$

$$= 4a^2e^2 + 4aer\cos\theta + r^2;$$

$$\therefore\ r = \frac{a(1 - e^2)}{1 + e\cos\theta}.$$

133. If C be the pole, $CP = r$, and $PCM = \theta$.

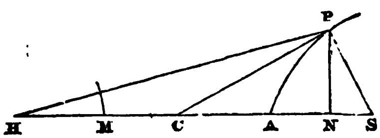
Then $x = r\cos\theta$, and $y = r\sin\theta$;

$$\therefore\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = r^2.\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) = 1;$$

$$\therefore\ r = \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} = \frac{b}{\sqrt{1 - e^2\cos^2\theta}}.$$

The Hyperbola.

134. If the difference between SP and PH be constant, the locus of P is the *hyperbola.*



Let the difference be $2a$; bisect SH in C. Take $CA = a = CM$, the curve passes through A.

$$\left.\begin{array}{l} CN = x \\ NP = y \end{array}\right\}. \quad \text{Let } CS = e\,.\,CA = ea, \text{ where } e > 1.$$

Then $HP^2 = HN^2 + NP^2 = (ea + x)^2 + y^2 = D_1^2$,

$$SP^2 = SN^2 + NP^2 = (ea - x)^2 + y^2 = D^2;$$

$$\therefore\ D_1^2 + D^2 = 2\,.\,(a^2e^2 + x^2 + y^2); \quad D_1^2 - D^2 = 4aex.$$

$$\text{Also } D_1 - D = 2a; \quad \therefore\ D_1 + D = 2ex;$$

$$\therefore\ D_1 = a + ex, \text{ and } D = ex - a;$$

$$\therefore\ 2a^2 + 2e^2x^2 = 2\,(a^2e^2 + x^2 + y^2);$$

$$\therefore\ y^2 = (e^2 - 1)\,.\,x^2 - (e^2 - 1)\,a^2 = (e^2 - 1)\,(x^2 - a^2)$$

$$= \frac{b^2}{a^2}\,.\,(x^2 - a^2); \text{ making } b^2 = a^2\,(e^2 - 1);$$

$$\therefore\ \frac{y^2}{b^2} - \frac{x^2}{a^2} = -1.$$

COR. 1. If A be the origin, and $AN = x_1$; $\therefore\ x = x_1 + a$.

$$\therefore\ x + a = x_1 + 2a; \quad x^2 - a^2 = x_1^2 + 2ax_1;$$

$$\text{and } y^2 = \frac{b^2}{a^2}\,(2ax_1 + x_1^2).$$

135. To find the polar equation, S being the pole.

$$SP = r, \quad \angle ASP = \theta.$$

Then $(2a + r)^2 = HP^2 = PN^2 + HN^2$

$$= PN^2 + (2CS - SN)^2$$

$$= r^2 \sin^2\theta + (2ae - r\cos\theta)^2;$$

$$\therefore\ 4a^2 + 4ar + r^2 = r^2 + 4a^2e^2 - 4aer\cos\theta;$$

$$\therefore\ r = \frac{a\,(e^2 - 1)}{1 + e\cos\theta}.$$

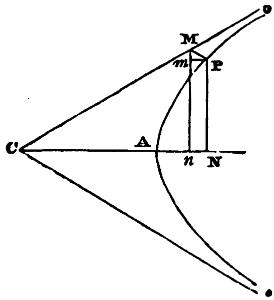
136. Let C be the pole; $CP = r$; $\angle ACP = \theta$.

$$\therefore\ x = r\cos\theta, \text{ and } y = r\sin\theta;$$

$$\therefore\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = r^2\left(\frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2}\right) = 1;$$

$$\therefore\ r = \frac{ab}{\sqrt{b^2\cos^2\theta - a^2\sin^2\theta}} = \frac{b}{\sqrt{e^2.\cos^2\theta - 1}}.$$

137. The asymptotes being the axes, and the centre the origin, find the equation to the hyperbola.



The asymptotes are lines, as CO and Co, drawn through the centre, making an angle $= \tan^{-1}\dfrac{b}{a}$ with the axis of the hyperbola.

$$CN = x, \quad CM = x_1,$$

and $OCA = oCA = \theta$,

$$NP = y, \quad MP = y_1.$$

Draw $Mn \perp$ to CAN, and $Pm \perp$ to Mn.

Since MP is parallel to Co, and Pm to CN, $\therefore \angle MPm = \theta$.

Now $x = Cn + nN = x_1 \cos\theta + y_1 \cos\theta = (x_1 + y_1)\cos\theta$,

$$y = Mn - Mm = x_1 \sin\theta - y_1 \sin\theta = (x_1 - y_1)\sin\theta;$$

$$\therefore \frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{(x_1+y_1)^2}{a^2}\cos^2\theta - \frac{(x_1-y_1)^2}{b^2}\sin^2\theta = 1.$$

$$\text{But } \tan\theta = \frac{b}{a}; \quad \therefore 1 + \tan^2\theta = \frac{1}{\cos^2\theta} = \frac{b^2+a^2}{a^2};$$

$$\therefore \frac{\cos^2\theta}{a^2} = \frac{1}{b^2+a^2}, \text{ and } \frac{\sin^2\theta}{b^2} = \frac{\cos^2\theta}{a^2} = \frac{1}{b^2+a^2};$$

$$\therefore \frac{(x_1+y_1)^2 - (x_1-y_1)^2}{b^2+a^2} = 1;$$

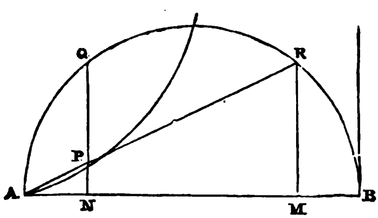
$$\text{i.e. } 4x_1y_1 = a^2 + b^2, \quad x_1y_1 = \frac{a^2+b^2}{4}.$$

Cor. If the hyperbola be rectangular,

$$b = a, \text{ and } x_1y_1 = \frac{a^2}{2}.$$

The Cissoid.

138. AQB is a semicircle. Take AN and BM equal. Draw the ordinates NQ, MR. Join AR cutting NQ in P. The locus of P is the cissoid.



$$AN = x, \quad NP = y,$$

$$AB = 2a.$$

Now $\frac{AN^2}{NP^2} = \frac{AM^2}{MR^2} = \frac{AM^2}{AM \cdot MB} = \frac{AM}{MB}$,

$$\text{or } \frac{x^2}{y^2} = \frac{2a-x}{x}; \quad \therefore y^2 = \frac{x^3}{2a-x}.$$

139. To find the Polar Equation.

$$AP = r, \quad \angle PAN = \theta, \quad x = r\cos\theta, \quad y = r\sin\theta,$$

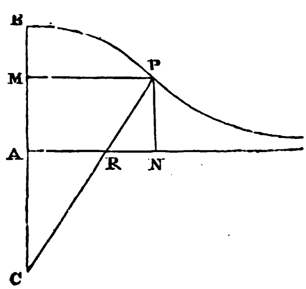
$$\frac{y^2}{x^2} = \frac{\sin^2\theta}{\cos^2\theta} = \frac{x}{2a-x} = \frac{r\cos\theta}{2a - r\cos\theta};$$

$$\therefore 2a\sin^2\theta = r\cos\theta\,(\sin^2\theta + \cos^2\theta);$$

$$\therefore r = 2a\frac{\sin\theta}{\cos\theta}\cdot\sin\theta = 2a\tan\theta\cdot\sin\theta.$$

The Conchoid of Nicomedes.

140. The line CP revolves round a fixed point C, cutting the line ARN: RP is always of the same length; then the point P will trace out the conchoid.



Let $CA = a$, $AM = x$,
$RP = AB = b$, $MP = y$.

$$\frac{MP^2}{CM^2} = \frac{AR^2}{CA^2} = \frac{RN^2}{NP^2} = \frac{RP^2 - NP^2}{NP^2};$$

$$\text{or } \frac{y^2}{(a+x)^2} = \frac{b^2-x^2}{x^2};$$

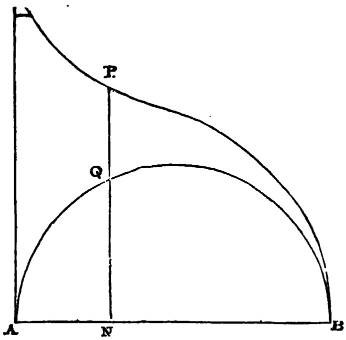
$$\therefore x^2y^2 = (a+x)^2(b^2 - x^2).$$

Cor. Let $CP = r$, $\angle PCM = \theta$,

$$r = CP = PR + CR = b + \frac{a}{\cos\theta}.$$

The Witch.

141. AQB is a semicircle, and NP is taken a fourth proportional to AN, AB, and NQ.

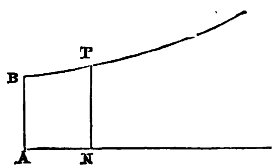


Then the locus of P is the Witch of Agnesi.

$$AN = x,$$
$$AB = 2a,$$
$$NP = y;$$
$$\therefore\ NQ = \sqrt{2ax - x^2},$$
$$x : 2a :: \sqrt{2ax - x^2} : y;$$
$$\therefore\ y = \frac{2a\sqrt{2ax - x^2}}{x} = 2a\sqrt{\frac{2a - x}{x}}.$$

142. The Logarithmic Curve.

In this curve, the abscissa is the logarithm of the ordinate, or if a be the base of the system, the equation to the curve is $y = a^x$;

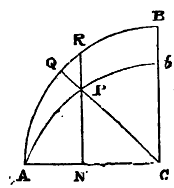


$$\therefore\ AB = a^0 = 1,$$

or the ordinate through the origin is always unity.

143. The Quadratrix of Dinostratus.

While the ordinate RN of the quadrant AQB moves uniformly from A to BC, the radius revolves from CA to CB, cutting RN in P; the locus of P is the curve required.



$$AN = x, \qquad CB = a,$$
$$NP = y, \qquad \angle QCA = \theta.$$

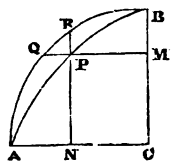
$$\text{Then } \theta : \frac{\pi}{2} :: x : a; \quad \therefore\ \theta = \frac{\pi x}{2a};$$

$$\frac{PN}{CN} = \frac{y}{a - x} = \tan\theta = \tan\frac{\pi x}{2a};$$

$$y = (a - x)\,.\tan\frac{\pi x}{2a}.$$

COR. When $x=a$; $y=Cb=\frac{2a}{\pi}$, which is a third proportional to the quadrant and radius.

144. If RN move as before, and a line as QPM parallel to AC move from AC, so that Q moves uniformly through AQ, the intersection P of RN and QM will trace the Quadratrix of Tschirnhausen.



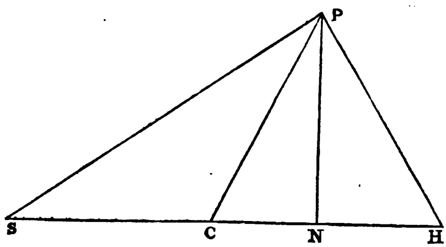
Here $AQ : AQB :: x : a$;

$$\therefore\ AQ=\frac{x}{a}\cdot\frac{\pi a}{2}=\frac{\pi x}{2};$$

$$\therefore\ y=a\,.\sin\left(\frac{AQ}{a}\right)=a\,.\sin\frac{\pi x}{2a}\text{ is the equation.}$$

The Lemniscata.

145. If SH be a straight line bisected in C, and if SP and HP revolve round S and H, and intersect in P, so that $SP\times HP=CS^2$, the locus of P is the Lemniscata.



$$CN=x;\quad NP=y;\quad CS=a,$$
$$SP=\sqrt{PN^2+SN^2}=\sqrt{y^2+(a+x)^2},$$
$$HP=\sqrt{PN^2+HN^2}=\sqrt{y^2+(a-x)^2};$$
$$\therefore\ \sqrt{y^2+(a+x)^2}\times\sqrt{y^2+(a-x)^2}=a^2;$$
$$\therefore\ \{(y^2+a^2+x^2)+2ax\}\{(y^2+a^2+x^2)-2ax\}=a^4;$$
$$\therefore\ (y^2+a^2+x^2)^2-4a^2x^2=a^4;$$
$$\therefore\ y^4+2x^2y^2+x^4=2a^2x^2-2a^2y^2;$$
$$\therefore\ (y^2+x^2)^2=2a^2(x^2-y^2).$$

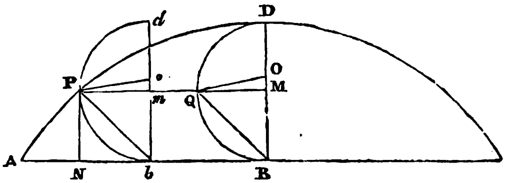
COR. If $CP=r$; and $\angle PCH=\theta$.

Then $x=r\cos\theta$; $y=r\sin\theta$; $x^2+y^2=r^2$;

$$\therefore\ r^4=2a^2r^2(\cos^2\theta-\sin^2\theta);\quad \therefore\ r^2=2a^2\cos 2\theta.$$

The Cycloid.

146. *The Cycloid* is described by a point in the circumference of a circle, which rolls along a horizontal line.



Let BQD be the circle, O the centre; and when its diameter is perpendicular to the horizontal line at A, let the point P, which generates the curve, also be at A.

Then Ab must = arc Pb, since each point of Pb has been in contact with each successive point of Ab.

147. Let $AN = x$, $\quad BD = 2a$,

$NP = y$, $\quad \angle QOB = \theta$;

$$\therefore\ x = Ab - Nb = a\theta - a\sin\theta = a(\theta - \sin\theta);$$

$$\therefore\ y = bm \qquad = a \text{ ver. } \sin\theta = a(1 - \cos\theta);$$

an algebraic equation cannot be found between x and y, but a differential equation may; for

$$x = a \text{ ver. } \sin^{-1}\frac{y}{a} - \sqrt{2ay - y^2};$$

$$\therefore\ \frac{dx}{dy} = \frac{a}{\sqrt{2ay - y^2}} - \frac{a - y}{\sqrt{2ay - y^2}} = \frac{y}{\sqrt{2ay - y^2}}.$$

148. To find the equation from the vertex D.

Let $DM = x$; $MP = y$; $\angle DOQ = \theta$.

Join Pb and QB, then these being equal and parallel,

$$PQ = Bb = AB - Ab = AB - Pb = DQ.$$

Then $y = PM = MQ + PQ = a\sin\theta + a\theta = a(\theta + \sin\theta)$,

$x = DM = a \text{ ver. } \sin\theta \qquad = a(1 - \cos\theta)$.

149. Since $x = a \text{ ver. } \sin\theta$; $\therefore\ \theta = \text{ver. } \sin^{-1}\frac{x}{a}$, and

$$a\sin\theta = \sqrt{2ax - x^2};$$

$$\therefore\ y = \sqrt{2ax - x^2} + a \text{ ver. } \sin^{-1}\frac{x}{a};$$

$$\therefore \frac{dy}{dx} = \frac{a-x}{\sqrt{2ax-x^2}} + \frac{a}{\sqrt{2ax-x^2}}$$

$$= \frac{2a-x}{\sqrt{2ax-x^2}} = \frac{\sqrt{2ax-x^2}}{x},$$

the equation most commonly used.

150. From (2°) may be derived a mechanical method of describing the cycloid; for the point P is found by drawing MP perpendicular to DB, and equal to the sum of the ordinate QM and the arc DQ of the circle.

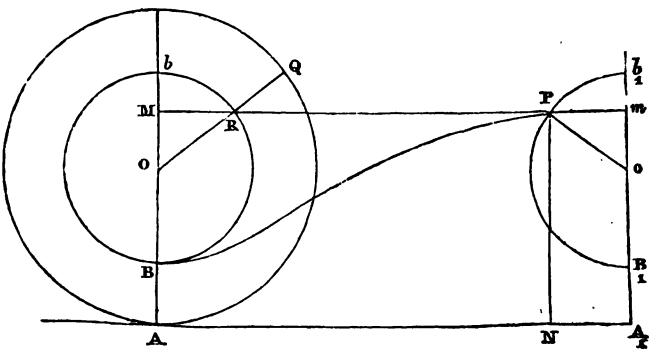
151. If we take MP equal to DQ only, then the locus of P is a curve called the Companion to the Cycloid: and

$$y = a\theta; \quad x = a(1-\cos\theta); \quad \frac{dy}{dx} = \frac{a}{\sqrt{2ax-x^2}}.$$

The Trochoid.

152. The trochoid is the curve traced by a point B in the circumference of the inner circle BRb, whilst the outer circle AQ rolls upon a horizontal line.

P a point in the trochoid. Through P draw a hori-



zontal line $MRPm$. Take O and o the centres of the circles.

Draw ORQ and oP.

Then $Pm = RM$, and $\angle AOQ = \angle A_1oP$.

Let $\left.\begin{array}{ll} OA = a, & AN = x \\ OB = b, & NP = y \end{array}\right\}$, $\angle AOR = \theta$.

Then it is obvious that arc $AQ = AA_1$;

$$\therefore\ x = AA_1 - NA_1 = a\theta - b\sin\theta,$$

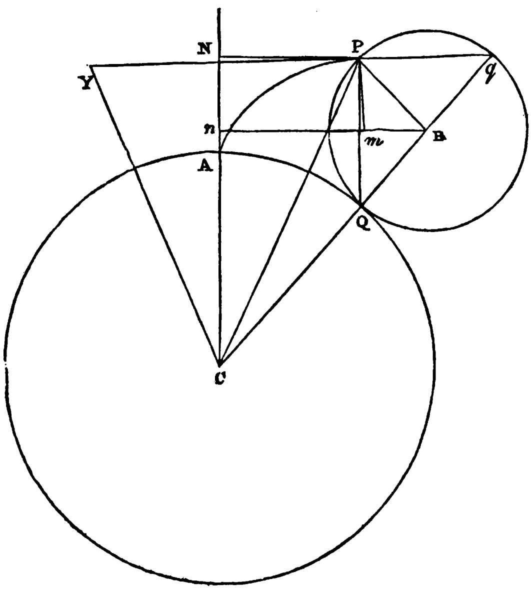
$$y = NP = oA_1 + om = a - b\cos\theta.$$

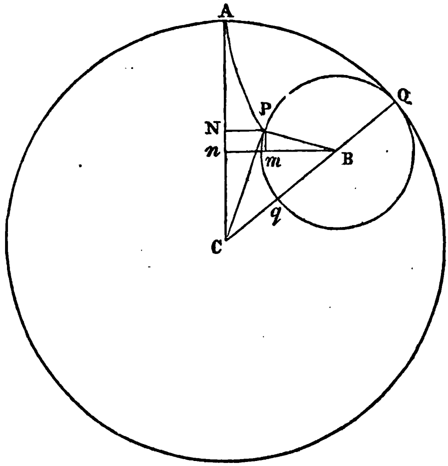
Let $e = \dfrac{b}{a}$; $\therefore\ x = a(\theta - e\sin\theta)$; $y = a(1 - e\cos\theta)$.

If $e = 1$; the trochoid becomes the cycloid.

The Epicycloid and Hypocycloid.

153. If one circle revolve upon another circle and in the same plane with it, the curve described by any point in the circumference of the revolving circle is called the Epicycloid; but if the revolving circle move within the other circle, the curve described by the point is called the Hypocycloid.





P the describing point of epicycloid.

$$CA = a; \quad CB = b; \quad \angle ACQ = \theta;$$
$$CN = x; \quad NP = y; \quad \angle PBQ = \phi;$$
$$\therefore \ x = Cn + Nn = (a+b)\cos\theta + b\sin PBm.$$
$$\text{But } PBm = PBQ - (90 - \theta) = \phi + \theta - 90.$$

Now $AQ = PQ$; $\therefore \ a\theta = b\phi$; $\therefore \ \phi = \dfrac{a\theta}{b}$;

$$\therefore \ x = (a+b)\cos\theta - b\cos\left(\frac{a+b}{b}.\theta\right);$$
$$y = Bn - Bm = (a+b)\sin\theta - b\sin\left(\frac{a+b}{b}.\theta\right).$$

154. If b be negative, the curve is the hypocycloid.

$$\therefore \ x = (a-b)\cos\theta + b\cos\left(\frac{a-b}{b}.\theta\right),$$
$$\text{and } y = (a-b)\sin\theta - b\sin\left(\frac{a-b}{b}.\theta\right),$$

Or thus $PBm = PBC - nBC = 180 - \dfrac{a\theta}{b} - (90 - \theta)$

$$= 90 - \frac{a-b}{b}.\theta,$$
$$x = (a-b)\cos\theta + b\sin PBm$$
$$= (a-b)\cos\theta + b\cos\left(\frac{a-b}{b}.\theta\right),$$

$$y = (a-b)\sin\theta - b\cos PBm$$
$$= (a-b)\sin\theta - b\sin\left(\frac{a-b}{b}\cdot\theta\right).$$

155. If the describing point P be not in the circumference, but within the revolving circle and at a distance b_1 from the centre, the curves described are then respectively called the Epitrochoid and Hypotrochoid: and

$$\left.\begin{aligned} x &= (a+b)\cos\theta - b_1\cos\left(\frac{a+b}{b}\cdot\theta\right) \\ y &= (a+b)\sin\theta - b_1\sin\left(\frac{a+b}{b}\cdot\theta\right) \end{aligned}\right\}\ldots\ldots\ldots\ldots(1).$$

$$\left.\begin{aligned} x &= (a-b)\cos\theta + b_1\cos\left(\frac{a-b}{b}\cdot\theta\right) \\ y &= (a-b)\sin\theta - b_1\sin\left(\frac{a-b}{b}\cdot\theta\right) \end{aligned}\right\}\ldots\ldots\ldots\ldots(2).$$

156. If in the epicycloid $a = b$,

$$x = a(2\cos\theta - \cos 2\theta),$$
$$y = a(2\sin\theta - \sin 2\theta);$$
$$x^2 + y^2 = a^2(4 - 4\cos\theta + 1);$$
$$\therefore\ x^2 + y^2 - a^2 = 4a^2(1 - \cos\theta).$$
$$\text{But } x = a(2\cos\theta - 2\cos^2\theta + 1);$$
$$\therefore\ x - a = 2a\cos\theta(1 - \cos\theta);$$
$$y = 2a\sin\theta(1 - \cos\theta);$$
$$\therefore\ (x-a)^2 + y^2 = 4a^2(1 - \cos\theta)^2.$$
$$\text{But } 16a^2(1 - \cos\theta)^2 = (x^2 + y^2 - a^2)^2;$$
$$\therefore\ 4a^2\{y^2 + (x-a)^2\} = (x^2 + y^2 - a^2)^2.$$

COR. If $x - a = r\cos\phi$, and $y = r\sin\phi$,

$$4a^2r^2 = (r^2 + 2ar\cos\phi)^2;$$
$$\therefore\ r = 2a(1 - \cos\phi);$$

the curve is called from its form the Cardioid.

157. To find the equation to the epicycloid in terms of the radius vector CP and the perpendicular on the tangent CY.

Produce CB to q; join PQ, Pq, then since for an instant the revolving circle turns on Q, the motion of the point P must be perpendicular to QP, and therefore in the direction Pq: hence qP produced is a tangent to the curve.

Produce qP, and draw $CY \perp$ to it, and make

$CP = r$; $CY = p$; then $\because CY$ is parallel to PQ,

$$\frac{PY^2}{CQ^2} = \frac{qY^2}{Cq^2}; \text{ also, } Cq = a + 2b;$$

$$\therefore \frac{r^2 - p^2}{a^2} = \frac{(a+2b)^2 - p^2}{(a+2b)^2} = \frac{c^2 - p^2}{c^2}; \text{ if } c = a + 2b;$$

$$\therefore r^2c^2 - p^2c^2 = a^2c^2 - a^2p^2;$$

$$\therefore p^2 = c^2.\left(\frac{r^2 - a^2}{c^2 - a^2}\right).$$

Cor. For the hypocycloid, $c = a - 2b$ is $< a$;

$$\therefore p^2 = a^2.\left(\frac{a^2 - r^2}{a^2 - c^2}\right).$$

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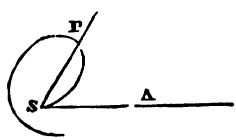
158. *Spirals.*

(1) The spiral of Archimedes. In this spiral the radius vector varies directly as the angle described,

or $r \propto \theta$; $\therefore r = a\theta$ is the equation.

Let a line revolve uniformly round S, while a point P moves uniformly from S along it, then P will trace the spiral of Archimedes.



Let $\angle ASP = \theta$, $SP = r$; and let c be the value of r when $\theta = 2\pi$;

$$\therefore r : c :: \theta : 2\pi;$$

$$\therefore r = \frac{c}{2\pi}\theta = a\theta \text{ by putting } a = \frac{c}{2\pi}.$$

(2) The logarithmic spiral. Here the a re described is the logarithm of the radius vector; its equation is $r = a^\theta$.

It is also called the equiangular spiral, since, as will be shewn, it cuts the radius at a constant angle.

(3) The hyperbolic spiral. In this spiral as the angle increases the radius vector decreases, and its equation is $r = \dfrac{a}{\theta}$.

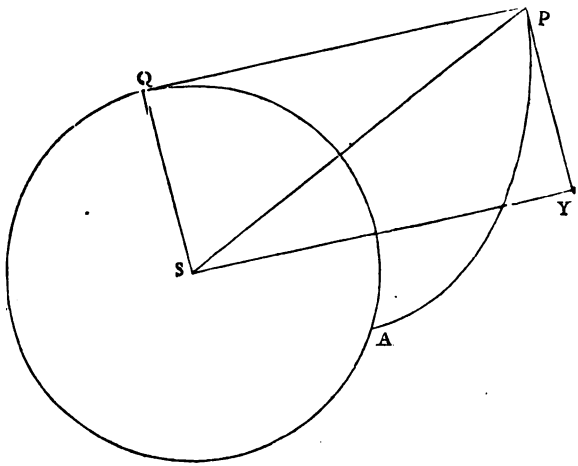
(4) The *lituus*, so called from its form; where $\theta = \dfrac{a^2}{r^2}$.

(5) The spiral of Archimedes, the hyperbolic, and the

lituus, are included under the general equation $r=a\theta^n$, as we shall see by putting $n=1$, -1, or $\frac{-1}{2}$.

The Involute of the Circle.

159. This curve is described by the extremity of a string which is unwound from the circumference of a circle.



A the point from which the string began to be unwrapped, *QP* the string once coincident with, and therefore $=AQ$; *PY* a tangent to *AP*, *SY* perpendicular to *PY*, join *SP*.

$$SP=r, \quad SY=p, \quad SQ=a;$$

$$\therefore\ SQ=PY=\sqrt{SP^2-SY^2};$$

$$\therefore\ a^2=r^2-p^2; \quad \therefore\ p^2=r^2-a^2, \text{ is the equation.}$$

Cor. If $\phi=\sec^{-1}\frac{r}{a}=PSQ$, and $\theta=\angle ASP$,

$$\theta+\phi=\frac{\sqrt{r^2-a^2}}{a}; \quad \therefore\ \theta=\frac{\sqrt{r^2-a^2}}{a}-\sec^{-1}\frac{r}{a};$$

$$\therefore\ a\frac{d\theta}{dr}=\frac{\sqrt{r^2-a^2}}{r}=\frac{SY}{SP}.$$

CHAPTER XI.

Tangents to Curves.

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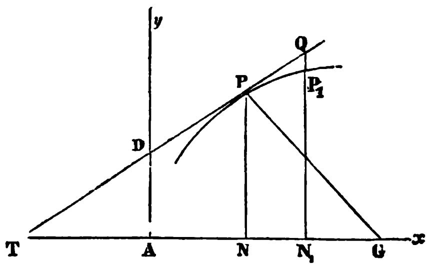
160. **Def.** A TANGENT to a curve is a straight line which has a point in common with the curve, but which, if produced, does not cut the curve in the neighbourhood of the point.

Let PP_1 be the curve, and let a line pass through the points P and P_1, and then revolve round P, so as to make P_1 continually approach to P; the position of the revolving line, when P_1 coincides with P, will be that of the tangent.



Let $AN = x$, Ax, Ay the axes of x and y.
$NP = y$, QPT the last position of the line.
$y = f(x)$ the equation to the curve.
$y_1 = ax_1 + b$ that of the line through P and P'.
$NN_1 = h$; $N_1P_1 = y'$; $AN_1 = x' = x + h$.

Then because the line passes through P and P_1,

$$y_1 - y = \frac{y'-y}{x'-x}\cdot(x_1 - x) \text{ is its equation.}$$

But $y' = f(x+h) = y + \frac{dy}{dx}h + Ph^2$;

$$\therefore \frac{y'-y}{x'-x} = \frac{\frac{dy}{dx}h + Ph^2}{h} = \frac{dy}{dx} + Ph.$$

Now since by the revolution of the line (supposed to pass through P_1 and P), P_1 approaches P; therefore h

decreases, and the right-hand member of the equation continually approaches $\frac{dy}{dx}$, and when P_1 actually coincides with P, $h=0$, and $\frac{y'-y}{x'-x}=\frac{dy}{dx}$; but then the line becomes a tangent,

$$\text{and } y_1-y=\frac{dy}{dx}(x_1-x) \text{ is the equation required.}$$

Cor. 1. The equation to the tangent may be written

$$y_1=\frac{dy}{dx}x_1+y-x\frac{dy}{dx},$$

which compared with the general equation to the straight line, $y=mx+b$, gives

$$m=\frac{dy}{dx}; \quad b=y-x\frac{dy}{dx},$$

which shew that the tangent is inclined to the axis of x at an angle whose tangent is $\frac{dy}{dx}$, and it cuts off from the axis of y a line $=y-x\frac{dy}{dx}$; from the former circumstance the Differential Calculus has been called the Method of Tangents.

Cor. 2. Hence $AD=y-x\frac{dy}{dx}$, and

$$\because \frac{AD}{AT}=\tan DTA=\frac{dy}{dx}; \quad \therefore AT=y\frac{dx}{dy}-x.$$

Cor. 3. We may deduce AD and AT from the general equation;

$$\text{let } x_1=0; \quad \therefore y_1=AD=y-x\frac{dy}{dx},$$

$$y_1=0; \quad \therefore -x_1=AT=y\frac{dx}{dy}-x;$$

or if these values be called y_0 and x_0,

$$y_0=y-x\frac{dy}{dx}; \quad x_0=x-y\frac{dx}{dy},$$

which are the parts cut off from the axes by the tangent.

Hence, find $\frac{dy}{dx}$ from the given equation, then AD and AT may be found; join TD, it produced is the tangent.

COR. 4. If the axes be oblique, we shall obtain similar results, but $\frac{dy}{dx}$ will not be the tangent of PTN, but the ratio of the sines of the angles the tangent makes with the axes.

COR. 5. The line NT is called the subtangent, and is useful in drawing the tangent,

$$\text{and } NT = AN + AT = x + y\frac{dx}{dy} - x = y\frac{dx}{dy}.$$

Hence to draw a tangent, find the value of NT. Join P, T, and we have the tangent required.

COR. 6. The length PT of the tangent

$$= \sqrt{PN^2 + NT^2} = \sqrt{y^2 + y^2\frac{dx^2}{dy^2}} = y\sqrt{1 + \frac{dx^2}{dy^2}}.$$

161. DEF. A line PG drawn from the point of contact P, perpendicularly to the tangent, and meeting the axis in G, is called the normal.

Since if $y = mx + b$ is the equation to a line;

$$\therefore\ y = -\frac{1}{m}x + b_1 \text{ is that of a perpendicular to it;}$$

$$\therefore \text{ since } y_1 = \frac{dy}{dx}x_1 + y - x\frac{dy}{dx} \text{ is equation to tangent;}$$

$$\therefore\ y_1 = -\frac{dx}{dy}x_1 + b_1 \text{ is that of the normal;}$$

$$\therefore\ y = -\frac{dx}{dy}x + b_1 \text{ since it passes through } P;$$

$$\therefore\ y_1 - y = -\frac{dx}{dy}(x_1 - x) \text{ is the equation required;}$$

and $\frac{dx}{dy}$ being found from the given equation to the curve, the normal may be drawn.

COR. 1. Hence $\tan PGx = -\frac{dx}{dy}$; and $\therefore \tan PGN = \frac{dx}{dy}$;

$$\text{also if } y_1 = 0;\quad x_1 = AG = y\frac{dy}{dx} + x;$$

$$\therefore\ NG, \text{ called the subnormal} = AG - AN = y\frac{dy}{dx}.$$

But NG may be found from the triangles NTP, PGN.

For $NG = \dfrac{NP^2}{NT} = \dfrac{y^2}{y \cdot \dfrac{dx}{dy}} = y\dfrac{dy}{dx}$.

Hence to draw a normal, find NG, and join PG.

Cor. 2. The length PG of the normal

$$= \sqrt{PN^2 + NG^2} = \sqrt{y^2 + y^2 \frac{dy^2}{dx^2}} = y\sqrt{1 + \frac{dy^2}{dx^2}}.$$

162. The normal is the shortest or longest line that can be drawn from a given point to a curve.

For if x_1 and y_1 be the co-ordinates of the given point,
x and y of a point in the curve,
u = distance between the points;

$$\therefore\ u^2 = (x_1 - x)^2 + (y_1 - y)^2 \text{ a maximum or minimum};$$

$$\therefore\ u\frac{du}{dx} = 0 = (x_1 - x) + (y_1 - y)\frac{dy}{dx} = 0;$$

$$\therefore\ y_1 - y = -\frac{dx}{dy}(x_1 - x),$$

which is, as we have just seen, the equation to the normal.

163. Find the length of the perpendicular from the origin upon the tangent, and the angle which the line from the origin to the point of contact makes with the tangent.

(1°) Since if δ be the perpendicular from the origin on a line

$$y = mx + b;\quad \delta = \frac{b}{\sqrt{m^2+1}} = \frac{y - mx}{\sqrt{m^2+1}};$$

$$\therefore\ \text{if } m = \frac{dy}{dx} = p;\quad \delta = \frac{y - px}{\sqrt{1+p^2}}.$$

(2°) Join AP, then if θ be the angle between the tangent and a line from the origin,

$$\text{or } = APT;\quad \angle APT = \angle PAN - \angle PTN,$$

$$\text{or } \theta = \tan^{-1}\frac{y}{x} - \tan^{-1}\frac{dy}{dx};$$

$$\therefore\ \tan\theta = \frac{\dfrac{y}{x} - \dfrac{dy}{dx}}{1 + \dfrac{ydy}{xdx}} = \frac{y - px}{x + py}.$$

164. It may be useful to collect these values in a table.

(1) $y_1 - y = \frac{dy}{dx}(x_1 - x)$, equation to tangent.

(2) $y_1 - y = -\frac{dx}{dy}(x_1 - x)$, equation to normal.

(3) Subtangent $NT = y\frac{dx}{dy}$.

(4) Subnormal $NG = y\frac{dy}{dx}$.

(5) Tangent $PT = y\sqrt{1 + \frac{dx^2}{dy^2}}$.

(6) Normal $PG = y\sqrt{1 + \frac{dy^2}{dx^2}}$.

(7) Perpendicular on tangent $= \frac{y - px}{\sqrt{1 + p^2}}$.

(8) Tangent $APT = \frac{y - px}{x + py}$.

(9) $AD = y_0 = y - x\frac{dy}{dx}$.

(10) $AT = -x_0 = y\frac{dx}{dy} - x$;

the first four of these formulas are the most important.

165. To find the tangent of the angle which the curve makes with the axis of x.

The angle which the tangent makes with the axis at the point of section will be the same that the curve makes. Find therefore the co-ordinates of the point of section, and substitute them in the expression for $\frac{dy}{dx}$, and the resulting value will be the tangent of the angle required.

Ex. 1. Let $y = \frac{x}{1 + x}$ be the equation to the curve.

Here if $x = 0$, $y = 0$; and $\therefore$ the origin is the point of section,

$$\text{and } \frac{dy}{dx} = \frac{1}{(1 + x)^2} = \frac{1}{1}, \text{ when } x = 0;$$

$$\therefore \tan\theta = 1 = \tan 45^\circ; \quad \therefore \theta = 45^\circ.$$

Ex. 2. Let the curve be the cycloid.

$$\text{Here } \frac{dy}{dx} = \frac{\sqrt{2a-y}}{\sqrt{y}} = \sqrt{\frac{2a}{y} - 1},$$

which is infinite if $y = 0$; or the curve cuts the axis at 90°.

Ex. 3. Let the curve be the circle, find AY and APT, A being the centre.

$$y^2 = a^2 - x^2, \quad \frac{dy}{dx} = p = -\frac{x}{y}; \quad 1 + p^2 = \frac{y^2 + x^2}{y^2} = \frac{a^2}{y^2},$$

$$y - px = y + \frac{x^2}{y} = \frac{a^2}{y}, \quad py + x = -x + x = 0;$$

$$\therefore \; AY = \frac{\frac{a^2}{y}}{\frac{a}{y}} = a; \quad \tan APT = \frac{\frac{a^2}{y}}{0} = \infty; \quad \therefore \; \angle APT = 90^\circ.$$

166. To draw a tangent through a given point.

Let α and β be the co-ordinates of the given point;

x and y be the co-ordinates of the curve;

$(y_1 - y) = \frac{dy}{dx}(x_1 - x)$ is the equation to the tangent.

But it passes through a point $y_1 = \beta$ and $x_1 = \alpha$;

$$\therefore \; \beta - y = \frac{dy}{dx}(\alpha - x);$$

from which, and the given equation to the curve, the point to which the tangent is to be drawn may be found.

167. To draw a tangent parallel to a given line.

Let $\tan^{-1} A =$ the angle the line makes with x;

$\therefore \; \frac{dy}{dx} = A$, since tangent and line are parallel;

and $y_1 - y = A \,.\, (x_1 - x)$ is the equation required.

If it pass through a given point, the co-ordinates of the point may be put for x_1 and y_1, and then from the given equation to the curve, and from that of the tangent, the point to which the tangent is to be drawn may be found.

168. To find the locus of the intersections of perpendiculars from the origin on the tangent, with the tangent.

Let $y = f(x)$ be the equation to the curve;

$$\therefore\ y_1 - y = \frac{dy}{dx}(x_1 - x) \text{ is the equation to the tangent.}$$

$$y_1 = -\frac{dx}{dy}x_1 \text{ is that of perpendicular from origin.}$$

Between these equations and $\frac{dy}{dx} = f'(x)$ eliminate y, x, and $\frac{dy}{dx}$; the resulting equation, containing y_1, x_1, and constant quantities, will be that of the curve required.

Prob. If δ be the length of the perpendicular from the origin upon the tangent, shew that $\delta = \sqrt{x_1 x + yy_1}$.

For $\because\ y_1 - y = \frac{dy}{dx}(x_1 - x)$........................(1),

and $y_1 = -\frac{dx}{dy}x_1$................................(2),

are simultaneous equations for the point of intersection;

$$\therefore\ (1)\times(2),\ y_1^2 - yy_1 = -(x_1^2 - xx_1);$$
$$\therefore\ y_1^2 + x_1^2 = \delta^2 = yy_1 + xx_1;$$
$$\therefore\ \delta = \sqrt{yy_1 + xx_1}.$$

169. Two curves whose equations are $y = f(x)$; and $y' = \phi(x')$ intersect, find the angle of intersection.

Let θ be the angle between the tangents at the point of intersection, it will therefore be the angle required;

$$\text{also } \theta = \tan^{-1}\frac{dy}{dx} - \tan^{-1}\frac{dy'}{dx'}.$$

$$\text{But } \frac{dy}{dx} = f'(x);\quad \frac{dy'}{dx'} = \phi'(x') = \phi'(x).$$

Since at point of intersection $x' = x$;

$$\therefore\ \tan\theta = \frac{f'(x) - \phi'(x)}{1 + f'(x)\cdot\phi'(x)}.$$

Prob. The vertex of a parabola is in the centre of a circle, and its focus bisects the radius, find the angle of intersection of circle and parabola.

Here $y^2 = 2ax$ (1), and $y^2 = a^2 - x^2$ (2), are the equations to the parabola and the circle.

From (1) $\frac{dy}{dx} = f'(x) = \frac{a}{y}$; (2) $\frac{dy}{dx} = \phi'(x) = -\frac{x}{y}$;

$$\therefore \tan\theta = \frac{\dfrac{a+x}{y}}{1-\dfrac{ax}{y^2}} = \frac{y(a+x)}{y^2-ax} = \frac{y(a+x)}{ax}.$$

But $2ax = a^2 - x^2$ at point of intersection;

$$\therefore x + a = a\sqrt{2}; \quad x = a(\sqrt{2}-1); \quad y = a\sqrt{2}\sqrt{\sqrt{2}-1};$$

$$\therefore \tan\theta = \frac{2\sqrt{\sqrt{2}-1}}{\sqrt{2}-1} = \frac{2}{\sqrt{\sqrt{2}-1}}.$$

Asymptotes.

170. Asymptotes are tangents to the curve at a point infinitely distant from the origin, and may be drawn, if the values of AD or AT, or of both remain finite, when either x or y, or x and y, are infinite.

Asymptotes may be thus constructed:

(1) If AD and AT be finite, join T, D, and the line produced is the asymptote.

(2) If AD be infinite, and AT finite, the asymptote is perpendicular to the axis of x, passing through T.

(3) If AD be infinite, and $AT = 0$, the asymptote coincides with the axis of y.

(4) If AD be finite, and AT infinite, the asymptote is parallel to the axis of x; and if $AD = 0$ is coincident with it.

Example. Draw an asymptote to the hyperbola.

Here $y = \dfrac{b}{a}\sqrt{2ax + x^2}$, and $\dfrac{dy}{dx} = \dfrac{b}{a}\,\dfrac{a+x}{\sqrt{2ax+x^2}}$,

$$AD = y - x\frac{dy}{dx} = \frac{b}{a}\cdot\left\{\sqrt{2ax+x^2} - \frac{(a+x)x}{\sqrt{2ax+x^2}}\right\} = \frac{bx}{\sqrt{2ax+x^2}}$$

$$= \frac{b}{\sqrt{1+\dfrac{2a}{x}}} = b \text{ if } x = \infty;$$

$$AT = y\frac{dx}{dy} - x = \frac{2ax+x^2}{a+x} - x = \frac{ax}{a+x} = \frac{a}{1+\dfrac{a}{x}} = a; \text{ if } x = \infty;$$

$$\therefore AT = \tfrac{1}{2}\text{ major-axis, or } T \text{ and } C \text{ coincide.}$$

Join CD; it produced, is the asymptote.

171. This method is frequently difficult of application, and the following is more generally useful.

If possible, let the equation to the curve be put under the form $y = Ax + B + \frac{C}{x} + \frac{D}{x^2} + \frac{E}{x^3} + \text{\&c.}$ then it is obvious, that as x increases, the terms after B decrease; and when x becomes infinitely great, they vanish, and the equation to the infinite branch of the curve is $y = Ax + B$.

But this is the equation to a straight line cutting the axis of y at a point $y = B$, and making an angle $= \tan^{-1} A$, with the axis of x. Hence it appears that the infinite branch of the curve is coincident with the line determined by the equation $y = Ax + B$;

$\therefore$ if $y = Ax + B + \frac{C}{x} + \frac{D}{x^2} + \text{\&c.}$ be the equation to a curve,

$y = Ax + B$ is the equation to the asymptote.

We may observe that this method will not apply to find the asymptotes that are parallel to the axis of y: since then A would be infinite: but asymptotes of this kind are discoverable by simple inspection of the equation. (See Example 9.)

Cor. 1. If the form of the expanded $f(x)$ be

$$y = Ax^2 + Bx + C + \frac{D}{x} + \frac{E}{x^2} + \text{\&c.}$$

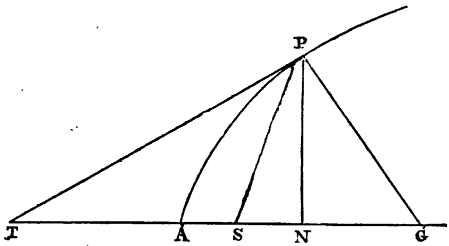
the asymptote is a parabolic curve, of which the equation is

$$y = Ax^2 + Bx + C.$$

Cor. 2. Since for every finite value of x, the ordinate of the curve $y = Ax + B + \frac{C}{x}$, differs less from that of the original curve, than the ordinate of the rectilinear asymptote, it is obvious that we may have a hyperbolic curve lying between the asymptote and the curve, and ultimately coincident with either; this may be called a hyperbolic asymptote.

Examples.

The Parabola.



(1) Here $y^2 = 4ax$; $\therefore \frac{dy}{dx} = \frac{2a}{y}$;

$$\therefore y\frac{dx}{dy} = NT = \frac{y^2}{2a} = 2x; \quad \therefore AT = AN.$$

Take $\therefore AT = AN$, join PT, PT is the tangent,

$$y\frac{dy}{dx} = 2a; \text{ or } NG = 2AS.$$

Take $\therefore NG = 2AS$; join PG, it is the normal;

equation to tangent is $y_1 - y = \frac{2a}{y}(x_1 - x)$;

$$\therefore yy_1 - y^2 = 2ax_1 - 2ax; \quad \therefore yy_1 = 2a(x_1 + x).$$

Cor. Since $yy_1 = 2a(x_1 + x)$ is equation to tangent;

$$\therefore y_1 = \frac{2a}{y}x_1 + \frac{2ax}{y}; \text{ make } \frac{2a}{y} = m;$$

$$\therefore \frac{2ax}{y} \text{ or } \frac{y}{2} = \frac{a}{m};$$

$$\therefore y_1 = mx_1 + \frac{a}{m};$$

an equation to the tangent which is often very convenient.

The equation to the normal is

$$y_1 - y = -\frac{y}{2a}(x_1 - x);$$

$$\therefore y_1 = -\frac{y}{2a}x_1 + y + \frac{yx}{2a}.$$

Make $\frac{-y}{2a}=m_1$; $\therefore \frac{y^2}{4a^2}=\frac{x}{a}=m_1^2$;

$$\therefore y+\frac{yx}{2a}=-2am_1-am_1^3;$$

$$\therefore y_1=m_1(x_1-2a-am_1^2);$$

the equation in terms of its inclination to the axis of x.

Ex. Two normals to a parabola intersect at right angles; find the locus of their intersection.

$$\therefore y+2am=m(x-am^2)\ \ldots\ldots\ldots\ldots (1),$$

$$\text{and } y-\frac{2a}{m}=-\frac{1}{m}\left(x-\frac{a}{m^2}\right)\ldots\ldots\ldots\ldots (2);$$

$$\therefore 2a\left(m+\frac{1}{m}\right)=x\left(m+\frac{1}{m}\right)-a\left(m^3+\frac{1}{m^3}\right);$$

$$\therefore x-3a=a\left(m-\frac{1}{m}\right)^2\ldots\ldots\ldots\ldots (3).$$

But from (2) $my-2a=-x+\frac{a}{m^2}$,

$$(1)\ \frac{1}{m}y+2a=x-am^2;$$

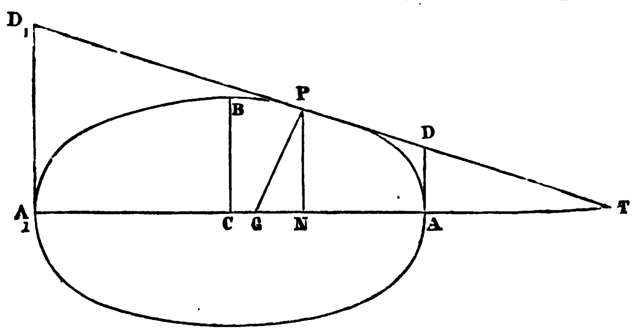
$$\therefore y\left(m+\frac{1}{m}\right)=a\left(\frac{1}{m^2}-m^2\right);$$

$$\therefore y=a\left(\frac{1}{m}-m\right)\ldots\ldots\ldots\ldots\ldots\ldots (4);$$

$$\therefore y^2=a^2\left(\frac{1}{m}-m\right)^2=a(x-3a),$$

the equation to a parabola.

(2) Find the equation to the tangent in the ellipse;



The centre being the origin,

$$\frac{y^2}{b^2}+\frac{x^2}{a^2}=1\,;\quad \therefore\ \frac{dy}{dx}=-\frac{b^2}{a^2}\cdot\frac{x}{y}\,;$$

$$\therefore\ y_1-y=-\frac{b^2}{a^2}\frac{x}{y}(x_1-x)\,;$$

$$\therefore\ a^2yy_1+b^2xx_1=a^2y^2+b^2x^2=a^2b^2.$$

COR. 1. $NT=-y\dfrac{dx}{dy}=\dfrac{a^2}{b^2}\cdot\dfrac{y^2}{x}=\dfrac{a^2-x^2}{x}$,

$$NG=-y\frac{dy}{dx}=\frac{b^2}{a^2}x.$$

COR. 2. $yy_1-y^2=\dfrac{-b^2}{a^2}xx_1+\dfrac{b^2}{a^2}x^2=-\dfrac{b^2}{a^2}xx_1+b^2-y^2$;

$$\therefore\ yy_1=\frac{b^2}{a^2}\cdot(a^2-xx_1),\ \text{or}\ y_1=\frac{b}{a}\cdot\frac{a^2-xx_1}{\sqrt{a^2-x^2}}.$$

$$\text{Let } y_1=0\,;\quad \therefore\ x_1=CT=\frac{a^2}{x}=\frac{CA^2}{CN}\,;$$

$$\therefore\ CT\times CN=CA^2,\ \text{(See } \textit{Conic Sections.}\text{)}$$

$$\text{and } NT=CT-CN=\frac{a^2-x^2}{x},$$

$$\text{or } NT\times CN=(a+x)(a-x)=A_1N\times AN.$$

COR. 3. Since $a^2yy_1+b^2xx_1=a^2b^2$; $\therefore\ y_1=-\dfrac{b^2x}{a^2y}x_1+\dfrac{b^2}{y}$.

$$\text{Let } m=-\frac{b^2x}{a^2y};\quad \therefore\ m^2a^2=\frac{b^4x^2}{a^2y^2};$$

$$\therefore\ m^2a^2+b^2=b^2.\left(\frac{b^2x^2+a^2y^2}{a^2y}\right)=\frac{a^2b^4}{a^2y^2}=\frac{b^4}{y^2};$$

$$\therefore\ y_1=mx_1+\sqrt{m^2a^2+b^2},$$

an equation to the tangent in terms of the angle it makes with the axis of x.

Ex. Find the locus of the intersection of pairs of tangents to an ellipse at right angles to each other.

$$y=mx+\sqrt{m^2a^2+b^2},\ \text{one tangent};$$

$$\therefore\ y=-\frac{x}{m}+\sqrt{\frac{a^2}{m^2}+b^2},\ \text{the other};$$

$$\therefore\ y-mx=\sqrt{m^2a^2+b^2},$$

$$my+x=\sqrt{m^2b^2+a^2};$$

squaring and adding,

$$(1+m^2)(y^2+x^2)=(1+m^2)(a^2+b^2);$$

$$\therefore\ y^2+x^2=a^2+b^2,$$

the equation to a circle, radius $=\sqrt{a^2+b^2}$.

COR. If $b=a$, the ellipse becomes a circle and radius $=a\sqrt{2}$, the chord of a quadrant.

(3) In the hyperbola of which the equation is

$$a^2y^2-b^2x^2=-a^2b^2,$$

the equation to the tangent is

$$a^2yy_1-b^2xx_1=-a^2b^2;$$

$$\therefore\ y_1=\frac{b^2x}{a^2y}x_1-\frac{b^2}{y}$$

$$=mx_1+\sqrt{m^2a^2-b^2};\ \text{ if } m=\frac{b^2x}{a^2y}.$$

Ex. Find the locus of the intersections of tangents to the hyperbola, and perpendiculars from the centre.

$$y_1=mx_1+\sqrt{m^2a^2-b^2},\ (1) \text{ is the tangent,}$$

$$y_1=-\frac{1}{m}x_1,\ (2) \text{ is the perpendicular;}$$

$\therefore\ m=-\dfrac{x_1}{y_1}$; substituting in (1), and omitting the suffix,

$$y=-\frac{x^2}{y}+\sqrt{\frac{a^2x^2}{y^2}-b^2};$$

$$\therefore\ (y^2+x^2)^2=a^2x^2-b^2y^2.$$

COR. Let $b=a$, or the hyperbola be the rectangular;

$$\therefore\ (y^2+x^2)^2=a^2(x^2-y^2),$$

the equation to the Lemniscata. In fact, the Lemniscata is commonly defined to be the locus of the intersection of the tangent to a rectangular hyperbola, with the perpendicular from the centre on the tangent.

(4) In the Cissoid, find the subtangent and subnormal, and equation to tangent.

$$\text{Here } y^2=\frac{x^3}{2a-x};$$

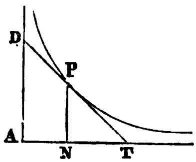
$$\therefore\ \text{subnormal }=y\frac{dy}{dx}=\frac{x^2(3a-x)}{(2a-x)^2};$$

$$\therefore \text{ subtangent } = y\frac{dx}{dy} = \frac{x(2a-x)}{3a-x}.$$

The equation to the tangent is

$$y_1 = \frac{\sqrt{x}}{(2a-x)^{\frac{3}{2}}}\cdot\{(3a-x)\,x_1 - ax\}.$$

(5) Rectangular hyperbola referred to the asymptotes.



Here $yx = \dfrac{a^2}{2}$; $\quad \therefore y = \dfrac{a^2}{2x}$,

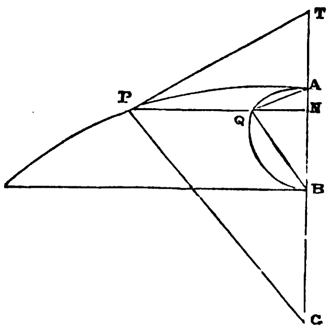
$$\frac{dy}{dx} = -\frac{a^2}{2}\cdot\frac{1}{x^2} = -\frac{y}{x};$$

$$\therefore\ y_1 - y = -\frac{y}{x}(x_1 - x),$$

$$xy_1 - xy = -yx_1 + yx; \quad xy_1 + yx_1 = 2yx = a^2;$$

$$x_1 = 0,\ y_0 = AD = \frac{a^2}{x}; \quad y_1 = 0, \quad x_0 = AT = \frac{a^2}{y}.$$

The $\triangle\, DAT = \dfrac{AT\,.\,AD}{2} = \dfrac{a^4}{2xy} = a^2$, which is constant.



Then $\dfrac{dy}{dx} = \dfrac{\sqrt{2ax-x^2}}{x}$;

$$\therefore\ y\,.\frac{dx}{dy} = \frac{y\,.\,x}{\sqrt{2ax-x^2}};$$

or $NT = \dfrac{NP\,.\,AN}{NQ}$; or

$NT : NP :: AN : NQ$;

and $\angle N$ is common to $\triangle$s ANQ, TPN; $\therefore$ they are similar, and $\angle PTN = \angle QAN$; $\therefore$ the tangent TP is parallel to the chord AQ.

Also since $\angle AQB$ is always $= 90$, PG is parallel to BQ.

(7) Draw an asymptote to the hyperbola.

$$y = \pm\frac{b}{a}\sqrt{2ax+x^2} = \pm\frac{b}{a}x\left(1+\frac{2a}{x}\right)^{\frac{1}{2}}$$

$$= \pm\frac{b}{a}x\left\{1 + \tfrac{1}{2}\frac{2a}{x} + \frac{\frac{1}{2}\cdot(\frac{1}{2}-1)}{1\,.\,2}\cdot\frac{4a^2}{x^2} + \frac{B}{x^3} + \&c.\right\}$$

$$= \pm\frac{b}{a}\cdot\left\{x + a - \frac{1}{2}\cdot\frac{a^2}{x} + \frac{B}{x^2} + \&c.\right\},$$

and therefore $y = \pm \frac{b}{a}(x+a)$ is the equation to two asymptotes; and since if $x=0$, $y=\pm b$; and if $y=0$, $x=-a$, both will pass through the centre, and be equally inclined to the axis of x.

(8) Draw the asymptote to the curve.

$$y^3 = x^3 + ax^2 = x^3\left(1+\frac{a}{x}\right);$$

$$\therefore\ y = x\left(1+\frac{a}{x}\right)^{\frac{1}{3}} = x\left\{1 + \tfrac{1}{3}\cdot\frac{a}{x} + \frac{A}{x^2} + \frac{B}{x^3} + \&c.\right\}$$

$$= x + \frac{a}{3} + \frac{A}{x} + \frac{B}{x^2} + \&c.;$$

$\therefore\ y = x + \frac{a}{3}$ is the equation to the asymptote which cuts the axis of x at an $\angle = 45^\circ$, and at a point $x = -\frac{a}{3}$.

(9) Let $y.(ax+b^2) = x^3$, draw the asymptotes.

$$y = \frac{x^3}{ax+b^2} = \frac{1}{a}\,\frac{x^3}{x+c} \text{ by putting } \frac{b^2}{a} = c$$

$$= \frac{1}{a}\cdot\frac{x^2}{1+\frac{c}{x}} = \frac{1}{a}x^2\left\{1 - \frac{c}{x} + \frac{c^2}{x^2} - \frac{c^3}{x^3} + \&c.\right\}$$

$$= \frac{x^2}{a} - \frac{cx}{a} + \frac{c^2}{a} - \frac{c^3}{ax} + \&c.;$$

$\therefore\ ay = x^2 - cx + c^2$ is the equation to the asymptotic curve.

Since $ay - \frac{3}{4}c^2 = x^2 - cx + \frac{c^2}{4} = \left(x - \frac{c}{2}\right)^2$,

or $\left(x-\frac{c}{2}\right)^2 = a\left(y - \frac{3}{4}\frac{c^2}{a}\right)$,

it shews that the curve is a parabola, the axis of which is perpendicular to the axis of x, and the vertex determined by making $x_1 = \frac{c}{2}$ and $y_1 = \frac{3}{4}\frac{c^2}{a}$; the latus rectum $= a$.

The curve has also a rectilinear asymptote parallel to the axis of y; for making $x = -c = -\frac{b^2}{a}$, y and $\frac{dy}{dx}$ are infinite; therefore an infinite ordinate at a distance $-\frac{b^2}{a}$ from the origin, will be a tangent to the curve.

172. When the equation between y and x cannot be solved with regard to y, it is frequently convenient to substitute zx for y; and then from the given equation to find x and y in terms of z; and z being so assumed as to make x and y infinite, the inclination of the asymptote, if any, to the axis of x is known (for $z=\frac{y}{x}=\tan$ inclination); and AD and AT being determined, the asymptote may be drawn.

(10) Let $ay^4-bx^4+c^3xy=0$; make $y=xz$;

$$\therefore\ ax^4z^4-bx^4+c^3x^2z=0;$$

$$\therefore\ x^2=\frac{c^3z}{b-az^4}=\infty,\ \text{if } az^4=b;\ \text{ or } z=\sqrt[4]{\frac{b}{a}};$$

$\therefore\ y=x\sqrt[4]{\frac{b}{a}}$ is the equation to the asymptote, since

$$AD=y-px=\frac{-2c^3xy}{4ay^3+c^3x}=0;\ \text{if } y=\infty;\ \text{and } x=\infty,$$

$$\text{and } AT=x-\frac{y}{p}=\frac{2c^3xy}{4ay^3+c^3x}=0;\ \text{if } y=\infty;\ \text{and } x=\infty.$$

PROB. Find that tangent line to a given curve which cuts off from the co-ordinate axes the greatest area.

$$\text{Area } =\tfrac{1}{2}x_0y_0=\tfrac{1}{2}(y-px)\left(x-\frac{y}{p}\right)$$

$$=-\tfrac{1}{2}\frac{(y-px)^2}{p};\ \text{ where } p=\frac{dy}{dx};$$

$$\therefore\ u=\frac{(y-px)^2}{p}\ \text{ a maximum};$$

$$\therefore\ \frac{du}{dx}=\frac{2}{p}(y-px)\left(\frac{dy}{dx}-p-x\frac{dp}{dx}\right)-\frac{(y-px)}{p^2}\frac{dp}{dx}=0;$$

$$\therefore\ -2px-(y-px)=0;$$

$$\therefore\ x=\tfrac{1}{2}\left(\frac{y-px}{p}\right)=\tfrac{1}{2}x_c.$$

Therefore also $y=\frac{1}{2}y_0$; and the tangent is bisected at the point of contact.

COR. Hence the least polygon of a given number of sides which will circumscribe a given oval, must have every side of the polygon bisected at the point of its contact with

the oval, since the interior area must be least, when the triangular spaces as in the preceding problem are greatest.

The axes are supposed rectangular; if oblique, the results are the same, but area $= \frac{1}{2} x_0 y_0 \times$ sine of inclination.

Examples.

(1) Let $y^n = a^{n-1}x$; $NT = nx$; $NG = \dfrac{y^2}{nx}$.

If $n = 2$, the curve is the parabola; $NT = 2x$, $NG = \dfrac{a}{2}$.

(2) Let the curve be the Witch:

$$NT = -\frac{2ax - x^2}{a}; \quad NG = -\frac{4a^3}{x^2}.$$

(3) The focus of a parabola is in the centre of a given circle, its vertex bisects the radius; find the point and angle of intersection of circle and parabola.

(4) Shew that the curve defined by $y^2 = 4ax$, intersects the curve defined by $y^2 = \dfrac{4}{27a}(x - 2a)^3$, at a point where $x = 8a$, and find the angle of intersection. Angle $= \tan^{-1} \dfrac{1}{\sqrt{2}}$.

(5) If $y^2 = 4a(x + a)$ be the equation to a parabola, the origin in the focus, shew that the points of intersection of the tangents, and perpendiculars from the focus, are determined by the equations

$$x_1 = -a, \text{ and } y_1 = \frac{y}{2}.$$

(6) The locus of the intersections of tangents to the parabola with perpendiculars from the vertex, is the cissoid.

(7) Find the length of the perpendicular drawn from the focus of an ellipse upon the tangent, and shew that the locus of their intersection is a circle, radius $= a$.

(8) Given two points A and B, find the locus of P when the angle PBA is double of the angle PAB, and draw an asymptote to the curve traced by P.

(9) Draw an asymptote to the curve defined by $y^3 + x^3 = 3axy$, and determine the points where the tangents are parallel to the two axes.

(10) Find the point and angle at which the curve $3y^2 = x(x + 2)^2$ cuts the axis. At the origin, angle $= 90$.

(11) Find the same when $y-2=(x-1)\sqrt{x-2}$, and the values of x and y when the tangent is perpendicular to the axis of x. (1) $y=0$; $x=3$. (2) $y=2$; $x=2$.

(12) If $y^3=3x^2-x^3$ draw an asymptote to the curve; find its greatest ordinate; and the angles the curve makes with the axis of x at the points, $x=0$, $x=2$ and $x=3$. Asymptote $y=-x+1$; abscissa $=2$; angles are 90, 0, 90.

(13) Draw the asymptotes, (1) when $y=\dfrac{x^3+ax^2+a^3}{x^2-a^2}$, (2) when $y^5=ax^4+mx^5$, and (3) when $y=a^x$.

(14) If in the ellipse, $\theta=\angle\, CPG$, and $l=$ the angle PG makes with the axis major; $\tan\theta=\dfrac{\tan l\,(a^2-b^2)}{a^2+b^2\tan^2 l}$.

(15) Shew that the curve whose equation is $\dfrac{y}{r}=\sin\dfrac{a}{r}$; where $y^2-2rx+x^2=0$; intersects the axis of x at points determined by $x=\dfrac{2a}{n\pi}$, n being any whole number.

(16) The normal to the curve defined by $y^2=4ax$, is a tangent to the curve in which $y^2=\dfrac{4}{27a}\cdot(x-2a)^3$.

(17) Draw a tangent to a circle, cutting the axis of x at 30°.

(18) In the conchoid, where $x^2y^2=(a+x)^2\cdot(b^2-x^2)$,

$$NT=-\frac{x.(a+x).(b^2-x^2)}{x^3+ab^2}.$$

(19) Draw a tangent to $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$; shew that the part of the tangent intercepted between the axes $=a$, and that perpendicular on tangent $=\sqrt[3]{axy}$.

(20) The centre of an ellipse is the vertex of a parabola, the major axis of ellipse is perpendicular to the axis of parabola, and the curves intersect at right angles, prove that major axis : minor axis :: $\sqrt{2}$: 1.

(21) If PY and QY be respectively tangents to corresponding arcs of a cycloid and the generating circle, the locus of Y is the involute of the circle.

(22) Find the angle of intersection of a rectangular hyperbola and a circle having the same centre, radius $=2a$;

angle $=\sin^{-1}\frac{1}{4}\sqrt{15}$.

(23) If TP and TQ be tangents to a parabola, and S be the focus, shew that

$$SP \,.\, SQ = ST^2.$$

(24) If $2y = c\,(e^{\frac{x}{c}} + e^{-\frac{x}{c}})$, (the equation to the catenary),

$$\text{the normal} = \frac{1}{c}\,y^2.$$

(25) If $\dfrac{1}{y^n} = \dfrac{1}{a^n} + \dfrac{1}{x^n}$ be the equation to a curve,

$$a^n \,.\, AT = x^{n+1}.$$

(26) If $\log\sqrt{x^2+y^2} = a\tan^{-1}.\dfrac{y}{x}$ be the equation to a curve, the angle in which it intersects the line drawn from the origin to the point of intersection is constant.

(27) If $\sqrt{y} = \sqrt{a} - \sqrt{x}$, find equation to the tangent and shew $AD + AT = a$.

(28) In the curve defined by the equation

$$y\,(1 + \log x) = x;\ NT : AT :: AN : NP.$$

(29) If $y = a\log(x^2 - a^2)$ be the equation to a curve

$$PT + NT = \frac{yx}{a}.$$

(30) Find that point in a parabola, at which a line from the vertex makes the greatest angle with the curve.

Ans. $x = 2a$.

(31) If A be the extremity of the diameter of a circle, PT a tangent, PN an ordinate, and AP a chord, prove that AP bisects the angle TPN.

(32) If $y^n - (a + bx)\,y^{n-1} + (c + ex + fx^2)\,y^{n-2} - \&\text{c.} = 0$, be the equation to a curve of n dimensions, prove that the sum of each ordinate divided by its respective subtangent, is a constant quantity.

(33) If C be the centre of a circle, AQ a chord, and let CNR be drawn cutting AQ in N, draw NP perpendicular to AQ and $= NR$; find the locus of P and draw its asymptote.

If $2c$ be the length of chord, a = radius of circle, origin centre of chord, and the chord be the axis of x, then

$$(y - a)^2 = x^2 + a^2 - c^2.$$

(34) ABD is a semicircle, centre C, and diameter AD, EF is a chord parallel to AD, CQR a radius cutting EF in Q, bisect QR in P; find the locus of P and the position of the asymptote. The curve is the conchoid.

CHAPTER XII.

The Differentials of the Areas and Lengths of Curves: of the Surfaces and Volumes of Solids of Revolution: Spirals.

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173. ONE of the applications of the Integral Calculus is to find the areas of curves, the lengths of their arcs, and the surfaces and contents of solids.

The solids of which we shall treat are called solids of revolution, since they are supposed to be generated by the revolution of a plane figure round a line, termed an axis. Hence every section perpendicular to the axis will be a circle, the radius of which is the revolving ordinate, and every section through the axis will reproduce the original area.

Considering the areas and lengths of curves, and the contents and surfaces of solids, to be functions of one of the quantities x or y, we can, by the Differential Calculus, find equations between the differential coefficients of these functions, and expressions containing x or y, by which we shall hereafter obtain the values of the functions themselves.

We shall find it useful first to establish the truth of the following Proposition.

174. If $A + Bx$, $A_1 + B_1x$, and $A + B_2x$ be three algebraical expressions taken in order of magnitude, viz., the first greater than the second, and the second greater than the third, whatever be the value of x; then shall $A_1 = A$.

For $(A + Bx) - (A + B_2x)$ is $> (A + Bx) - (A_1 + B_1x)$;

$$\therefore \text{ if } (A + Bx) - (A + B_2x) = 0, \text{ or } \frac{A + Bx}{A + B_2x} = 1;$$

$$\therefore \textit{à fortiori} \text{ will } \frac{A + Bx}{A_1 + B_1x} = 1.$$

But as x decreases $\dfrac{A + Bx}{A + B_2x}$ approaches $\dfrac{A}{A}$ or 1; and when x is diminished without limit it actually equals unity, and

$$\therefore \frac{A+Bx}{A_1+B_1x} \text{ which becomes } \frac{A}{A_1},$$

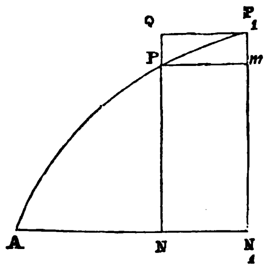
by the continued diminution of x, also is equal to unity;

$$\therefore \frac{A}{A_1}=1, \text{ or } A_1=A,$$

and since A_1 and A are independent of x, if they are once equal they are always so.

Area of a Curve.

175. Let AP be a curve, $y=f(x)$, the equation to it, where $AN=x$, $NP=y$; and let $A=$area ANP.



Then $\dfrac{dA}{dx}=y$.

Let $NN_1=h$. Complete the parallelograms QN_1 and PN_1.

Then the area P_1PNN_1 is $>\square PN_1$, $<\square QN_1$.........(1).

Now A depends upon x; for as x changes, A changes;

$$\therefore A=ANP=\phi(x); \text{ and } \therefore AN_1P_1=\phi(x+h);$$

$$\therefore PP_1N_1N=\phi(x+h)-\phi(x)=\frac{dA}{dx}h+\frac{d^2A}{dx^2}\frac{h^2}{1.2}+\&c.$$

$$\text{and } \square PN_1=yh.$$

$\square QN_1=h\times P_1N_1=h.f(x+h)=h\{y+ph+Ph^2\}$; $p=\dfrac{dy}{dx}$;

therefore, dividing by h, we have by (1),

$$\frac{dA}{dx}+\frac{d^2A}{dx^2}\frac{h}{1.2}+\&c.>y<y+ph+Ph^2;$$

$$\text{i.e. } y+ph+Ph^2,\ \frac{dA}{dx}+\frac{d^2A}{dx^2}\frac{h}{1.2}+\&c. \text{ and } y$$

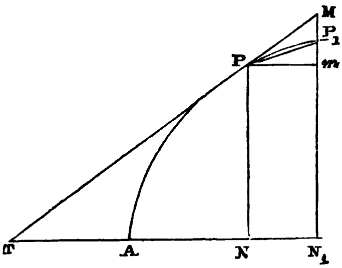
are in order of magnitude; whence, by the Lemma,

$$\frac{dA}{dx}=y.$$

Length of a Curve.

176. If s = length of the curve AP;

$$\frac{ds}{dx} = \sqrt{1 + \frac{dy^2}{dx^2}}.$$



Draw tangent PM, and chord PP_1.

Then arc PP_1 > chord $PP_1 < PM + MP_1$.

But arc $PP_1 = AP_1 - AP$

$$= \phi(x+h) - \phi(x) = \frac{ds}{dx}h + \frac{d^2s}{dx^2}\frac{h^2}{1.2} + \&c.$$

$$\text{chord } PP_1 = \sqrt{Pm^2 + (P_1m)^2}{}^* = \sqrt{h^2 + (ph + Ph^2)^2}$$

$$= h\sqrt{(1+p^2) + 2Pph + P^2h^2},$$

$$PM = \sqrt{Pm^2 + Mm^2}{}^* = \sqrt{h^2 + p^2h^2} = h\sqrt{1+p^2},$$

$$MP_1 = MN_1 - N_1P_1 = (y + ph) - (y + ph\ Ph^2) = -Ph^2;$$

whence, dividing by h,

$$\frac{ds}{dx} + \frac{d^2s}{dx^2}\cdot\frac{h}{1.2} + \&c. > \sqrt{1 + p^2 + 2Pph + P^2h^2} < \sqrt{1+p^2} - Ph$$

$$> \sqrt{1+p^2} + \frac{Pp}{\sqrt{1+p^2}}h + \&c. < \sqrt{1+p^2} - Ph;$$

$$\therefore \frac{ds}{dx} = \sqrt{1+p^2} = \sqrt{1 + \frac{dy^2}{dx^2}}.$$

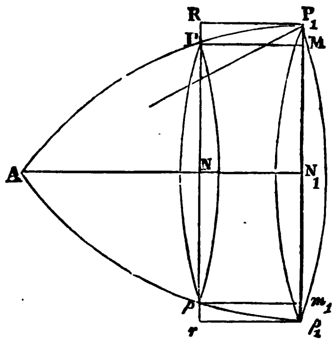
Volume of Solid.

177. If V be the volume of a solid of revolution APp,

$$\frac{dV}{dx} = \pi y^2.$$

* $P_1m = P_1N_1 - PN = y + ph + Ph^2 - y = ph + Ph^2$,

$Mm = Pm \cdot \tan MPm = h \cdot \tan PTN = h \cdot \frac{dy}{dx}$.



Let $\left.\begin{array}{r} AN = x \\ NP = y \\ NN_1 = h \end{array}\right\}$; $\therefore AP_1p_1 = f(x+h) = V + \frac{dV}{dx}h + \frac{d^2V}{dx^2}\frac{h^2}{1.2} + \&c.$

Then the solid Ppp_1P_1 is > cylinder PMm_1p,

$<$ cylinder RP_1p_1r;

$$\text{i.e. } \frac{dV}{dx}h + \frac{d^2V}{dx^2}\frac{h^2}{1.2} + \&c. > \pi y^2 h,$$

$$< \pi(y + ph + Ph^2)^2 h,$$

$$\text{or } \frac{dV}{dx} + \frac{d^2V}{dx^2}\frac{h}{1.2} + \&c. > \pi y^2 < \pi(y + ph + Ph^2)^2,$$

$$\text{or } > \pi y^2 < \pi y^2 + 2\pi pyh + \&c.$$

$$\text{whence } \frac{dV}{dx} = \pi y^2.$$

PROB. The surface of a truncated cone, of which the radii of the greater and smaller ends are a, b, and the slant side s, $= \pi s(a+b)$.

Let l = length of cone, radius of the base $= a$,

$l_1 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = b$;

therefore, surface of frustum

$$= \pi la - \pi l_1 b = \pi\{sa + l_1(a-b)\},$$

$$\text{but } l \text{ or } l_1 + s : l_1 :: a : b;$$

$$\therefore s : l_1 :: a - b : b;$$

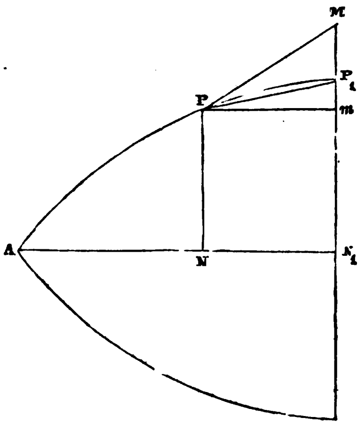
$$\therefore sb = l_1(a-b);$$

$$\therefore \text{surface of frustum} = \pi s.(a+b).$$

Surface of Solid.

178. If S = surface of the solid of revolution APp,

$$\frac{dS}{dx} = 2\pi y \sqrt{1 + \frac{dy^2}{dx^2}} = 2\pi y \cdot \frac{ds}{dx}.$$



$$\left.\begin{array}{r} AN = x \\ NP = y \\ AP = s \\ NN_1 = h \end{array}\right\}.$$ Draw the tangent PM, and chord PP_1.

Then, surface generated by arc PP_1, will be

$>$ than that by chord PP_1, $<$ by PM and MP_1.

Now chords PP_1 and PM generate truncated cones, of which the surfaces respectively are

$$\pi \{PN + P_1N_1\} PP_1, \text{ and } \pi \{PN + MN_1\} PM;$$

MP_1 generates a circular zone $= \pi (MN_1^2 - N_1P_1^2)$; the surface by arc $PP_1 = \frac{dS}{dx} h + \frac{d^2S}{dx^2} \frac{h^2}{1.2} + \&c.$

But $\{PN + P_1N_1\} PP_1$

$$= (2y + ph + Ph^2) \cdot \sqrt{h^2 + (ph + Ph^2)^2}$$

$$= (2y + ph + Ph^2) h \sqrt{1 + p^2 + Mh}, \text{ suppose};$$

$$(PN + MN_1) PM = (2y + ph) h \sqrt{1 + p^2},$$

$$MN_1^2 - N_1P_1^2 = - Ph^2 . (2y + 2ph + Ph^2) = - Nh^2;$$

$$\therefore\ \frac{dS}{dx}+\frac{d^2S}{dx^2}\,\frac{h}{1.2}+\&\text{c.}>\pi(2y+ph+Ph^2)\sqrt{1+p^2}+Mh$$
$$<\pi(2y+ph)\sqrt{1+p^2}-\pi Nh$$
$$>2\pi y\sqrt{1+p^2}+Mh+\text{terms involving } h,$$
$$<2\pi y\sqrt{1+p^2}+\pi ph\sqrt{1+p^2}-\pi Nh$$
$$>2\pi y\sqrt{1+p^2}+\pi\frac{yMh}{\sqrt{1+p^2}}+\&\text{c.}$$
$$<2\pi y\sqrt{1+p^2}+\pi ph\sqrt{1+p^2}-\pi Nh;$$
$$\therefore\ \frac{dS}{dx}=2\pi y\sqrt{1+p^2}=2\pi y\sqrt{1+\frac{dy^2}{dx^2}}=2\pi y\frac{ds}{dx}.$$

Cor. 1. Hence if A = area of ANP; and δA be the differential of the area, and δx of the abscissa; δx being very small,

$$\frac{\delta A}{\delta x}=\frac{dA}{dx}=y;\quad \therefore\ \delta A=y\delta x,$$

or the differential of the area equals the rectangle of the ordinate, and the increment of the corresponding abscissa.

Cor. 2. If δs be the increment of the arc AP; corresponding to the increments δx and δy of x and y;

$$\text{then } \because\ \frac{\delta s}{\delta x}=\frac{ds}{dx}=\sqrt{1+\frac{dy^2}{dx^2}}=\sqrt{1+\frac{\delta y^2}{\delta x^2}};$$
$$\therefore\ \delta s=\sqrt{\delta y^2+\delta x^2};$$

or the increment of the arc is the hypothenuse of a right-angled triangle, δx and δy being the two other sides.

Hence the chord PP_1 = arc PP_1 ultimately.

Cor. 3. If α be the angle which the tangent makes with x;

$$\therefore\ \frac{\delta s}{\delta x}=\sec\alpha;\quad \delta s=\delta x\,.\sec\alpha.$$

Cor. 4. In the same manner, if δV and δS be the differentials of the volume V, and of the surface S of a solid,

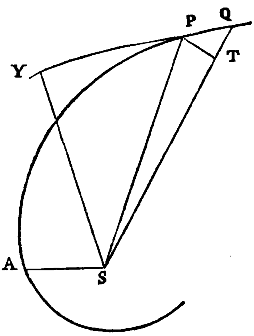
$$\frac{\delta V}{\delta x}=\pi y^2;\quad \therefore\ \delta V=\pi y^2\,.\,\delta x,$$
$$\text{and } \frac{\delta S}{dx}=2\pi y\,.\sqrt{1+\frac{dy^2}{dx^2}}=2\pi y\,.\frac{\delta s}{\delta x};$$
$$\therefore\ \delta S=2\pi y\,.\,\delta s;$$

hence it appears that the differential of a solid is a cylinder, base πy^2, and altitude δx: and that the differential of a surface is the convex surface of a cylinder, the circumference of whose base is $2\pi y$, and altitude δs.

Spirals.

179. The expressions just obtained, and those of the preceding Chapter, are only applicable, when the equation to the curve is known in terms of the rectangular co-ordinates; we shall now find corresponding expressions for the perpendicular upon the tangent, the area and length of a curve, when referred to polar co-ordinates; that is, when $r=f(\theta)$, or $p=f(r)$, p being the perpendicular on the tangent, r the radius vector, and θ the angle traced out by r.

180. To find the differential of the area of a Spiral.



Let area $ASP=A$; $SP=r$,

$\angle ASP=\theta$; $SY=p$.

Draw SQ very near to SP, and draw $PT \perp SQ$.

Then area $PSQ=\delta A$,

$\angle PSQ=\delta\theta$; $QT=\delta r$;

now PT is ultimately = a circular arc;

$$\therefore \delta A=\tfrac{1}{2}SQ\times PT \text{ very nearly}$$
$$=\tfrac{1}{2}(r+\delta r)\,.\,r\delta\theta \text{ very nearly};$$
$$\therefore \frac{\delta A}{\delta\theta}=\tfrac{1}{2}r^2+\tfrac{1}{2}r\delta r.$$

But as Q approaches P, δr continually diminishes, and ultimately vanishes; but then $\dfrac{\delta A}{\delta\theta}=\dfrac{dA}{d\theta}$,

$$\text{and } \therefore \frac{dA}{d\theta}=\tfrac{1}{2}r^2.$$

181. To find the differential of the length of a spiral.

Let $AP=s$; $\therefore PQ=\delta s$,

and $(\text{chord } PQ)^2=PT^2+QT^2=r^2.\delta\theta^2+(\delta r)^2$;

$$\therefore \frac{(\text{chord } PQ)^2}{\delta\theta^2} = \frac{\delta s^2}{\delta\theta^2} = r^2 + \left(\frac{\delta r}{\delta\theta}\right)^2,$$

$$\text{whence } \frac{ds}{d\theta} = \sqrt{r^2 + \frac{dr^2}{d\theta^2}}.$$

182. When $\theta = f(r)$, to find $\frac{d\theta}{dr}$.

From similar triangles, SPY and PQT,

$$\frac{PT}{QT} = \frac{SY}{PY}; \quad r \cdot \frac{\delta\theta}{\delta r} = \frac{p}{\sqrt{r^2 - p^2}};$$

$$\therefore \frac{d\theta}{dr} = \frac{p}{r\sqrt{r^2 - p^2}} \quad \ldots\ldots\ldots\ldots\ldots (1).$$

183. When $s = f(r)$, to find $\frac{ds}{dr}$.

$$\text{And } \frac{PQ}{QT} = \frac{\delta s}{\delta r} = \frac{SP}{PY} = \frac{r}{\sqrt{r^2 - p^2}};$$

$$\therefore \frac{ds}{dr} = \frac{r}{\sqrt{r^2 - p^2}}.$$

184. To find an expression for the perpendicular on the tangent,

$$\because \frac{d\theta}{dr} = \frac{p}{r\sqrt{r^2 - p^2}};$$

$$\therefore \frac{dr^2}{r^2 d\theta^2} = \frac{r^2 - p^2}{p^2} = \frac{r^2}{p^2} - 1;$$

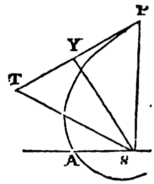
$$\therefore \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \cdot \frac{dr^2}{d\theta^2}.$$

$$\text{Let } u = \frac{1}{r}; \quad \therefore \frac{du}{d\theta} = -\frac{1}{r^2} \frac{dr}{d\theta};$$

$$\therefore \frac{1}{p^2} = u^2 + \frac{du^2}{d\theta^2},$$

whence given an equation between r or $\frac{1}{u}$ and θ, an equation between p and r may be found.

185. To draw a tangent to a spiral.



P the point to which the tangent is to be drawn. S the pole. Join SP. Suppose PT to be the tangent. Draw $SY \perp PT$, and $ST \perp PS$, then ST is called the sub-tangent.

$$\text{And } ST = SP \cdot \frac{SY}{PY} = \frac{pr}{\sqrt{r^2 - p^2}} = \text{also } r^2 \cdot \frac{d\theta}{dr}.$$

Find therefore from the equation to the spiral $\frac{pr}{\sqrt{r^2-p^2}}$, or $r^2 \cdot \frac{d\theta}{dr}$, according as the equation is $p = f(r)$, or $\theta = f(r)$. Draw ST perpendicular to SP and equal to either of these values. Join TP, it is the tangent.

Cor. Since $ST = \pm r^2 \cdot \frac{d\theta}{dr} = \mp \frac{d\theta}{du}$; $\therefore \frac{1}{ST^2} = \left(\frac{du}{d\theta}\right)^2$;

$$\therefore \frac{1}{SY^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{1}{SP^2} + \frac{1}{ST^2}.$$

186. Asymptotes to Spirals.

If ST remain finite when SP is infinite, a tangent may be drawn which will touch the curve at a point infinitely distant from S, and is therefore an asymptote. And since those lines are said to be parallel which coincide only at an infinite distance; the asymptote must be drawn parallel to the infinite line SP.

Hence to construct; find θ and $r^2 \cdot \frac{d\theta}{dr}$ when r is infinite. Draw SP at the angle thus found, ST perpendicular to SP, and TP parallel to the infinite radius vector, TP produced is the asymptote.

187. Asymptotic circle.

If in the equation $\theta = f(r)$, θ becomes infinite when $r = a$; but impossible if r be $> a$; then if we describe a circle with radius a, the spiral will make an infinite number of revolutions within the circle, and constantly approach the circumference, without exactly reaching it. In this case, the circle is called an exterior asymptotic circle. But if $r = a$ make θ infinite, and $r < a$, make θ impossible, the

without the circle *to which* ircle is an interior asymptote.

uation to the curve which is the tangent and the perpen- the locus of Y.

be the perpendicular from S ced by Y.

. See fig. Art. 185;

$$d\phi = \frac{p_1 dp}{p\sqrt{p^2 - p_1^2}}.$$

$$= \theta - \cos^{-1}.\frac{p}{r};$$

$$\frac{pdr}{r^2 - p^2} + \frac{rdp - pdr}{r\sqrt{r^2 - p^2}}$$

$$\frac{rdp}{r^2 - p^2};$$

$$\frac{p}{{}^2 - p^2}; \quad \therefore \frac{p^2}{p_1^2} = \frac{r^2}{p^2};$$

ind $p_1 = \dfrac{p^2}{r}$;

$= f(p) = f(r_1)$,

$_1) = r_1^2$,

is required.

equiangular where $p = mr$;

$$\frac{r_1^2}{r} = \frac{mr_1^2}{r_1} = mr_1,$$

Examples.

EXAMPLE. Find the value of p in the Conic Sections.

$$r=\frac{m}{1+e\cos\theta}, \text{ where } m=\tfrac{1}{2} \text{ latus rectum};$$

$$\therefore\ u=\frac{1}{m}+\frac{e}{m}.\cos\theta;\quad \frac{du}{d\theta}=-\frac{e}{m}.\sin\theta;$$

$$\therefore\ u^2+\frac{du^2}{d\theta^2}=\frac{1}{m^2}\{1+2e\cos\theta+e^2\}$$

$$=\frac{1}{m^2}(2mu-1+e^2)$$

$$=\frac{1}{m^2}.\left\{\frac{2m-r(1-e^2)}{r}\right\};$$

$$\therefore\ p^2=\frac{m^2.r}{2m-r(1-e^2)}.$$

(1) In parabola, $e=1$; $\therefore\ p^2=\frac{mr}{2}$, and $m=2SA$;

$$\therefore\ SY^2=SP.SA.$$

(2) In ellipse, $e<1$; $m=\frac{b^2}{a}$; $1-e^2=\frac{b^2}{a^2}$;

$$\therefore\ p^2=\frac{\frac{b^4}{a^2}.r}{2\frac{b^2}{a}-\frac{b^2}{a^2}.r}=\frac{b^2r}{2a-r}.$$

(3) In hyperbola, $e^2>1$; $e^2-1=\frac{b^2}{a^2}$;

$$\therefore\ p^2=\frac{m^2r}{2m+r(e^2-1)}=\frac{b^2r}{2a+r};$$

and therefore in ellipse and hyperbola, $SY^2=\frac{BC^2.SP}{HP}$.

(4) Find the equation between p and r, when $\theta=\frac{a^n}{r^n}$,

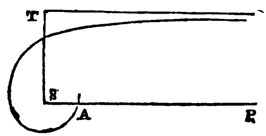
$$\theta=\frac{a^n}{r^n}=a^nu^n;\quad \therefore\ u^n=\frac{\theta}{a^n};$$

$$\therefore\ \frac{du}{d\theta}=\frac{1}{na^n.u^{n-1}}=\frac{r^{n-1}}{b^n} \text{ by substitution};$$

$$\therefore u^2 + \frac{du^2}{d\theta^2} = \frac{1}{r^2} + \frac{r^{2n-2}}{b^{2n}} = \frac{1}{r^2}\left\{\frac{b^{2n}+r^{2n}}{b^{2n}}\right\};$$

$$\therefore p = \frac{b^n . r}{\sqrt{b^{2n}+r^{2n}}}.$$

(5) Draw a tangent and asymptote to the spiral; where $\theta = \frac{a}{r} = au$; $\therefore \frac{1}{ST} = \frac{du}{d\theta} = \frac{1}{a}$; $\therefore ST = a$; or the locus of T is a circle radius $= a$.



Since ST is constant, and $\theta = 0$ when $r = \infty$. Produce SA indefinitely. Draw $ST \perp SA$ and $= a$. Then a line from T parallel to ST will be the asymptote required.

(6) Let $r = a^\theta$, the equation to the logarithmic spiral;

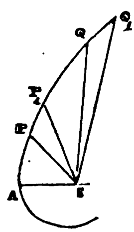
$$\therefore \frac{dr}{d\theta} = Aa^\theta = Ar; \quad \therefore \frac{d\theta}{dr} \text{ or } \frac{p}{r\sqrt{r^2-p^2}} = \frac{1}{Ar};$$

$$\therefore \frac{p}{\sqrt{r^2-p^2}} = \frac{1}{A}; \quad \therefore \frac{r^2}{p^2} = 1 + A^2;$$

$$\therefore \frac{p}{r} = \sin SPY = \frac{1}{\sqrt{1+A^2}} = m; \quad \therefore p = mr.$$

Since $\angle SPY$ is constant, the spiral is called the equiangular.

COR. 1. The radii including equal angles are proportionals.



Let SP and SP_1 include an $\angle \alpha$, and SQ and SQ_1 include the same angle.

Let $\angle ASP = \theta$, and $\angle ASQ = \phi$;

$$\therefore SP = a^\theta, \qquad SQ = a^\phi,$$

$$SP_1 = a^{\theta+\alpha}, \qquad SQ_1 = a^{\phi+\alpha};$$

$$\therefore \frac{SP_1}{SP} = a^\alpha, \text{ and } \frac{SQ_1}{SQ} = a^\alpha;$$

$$\therefore \frac{SP}{SP_1} = \frac{SQ}{SQ_1}, \text{ or } SP : SP_1 :: SQ : SQ_1.$$

(7) If $r = a(1 + \cos\theta)$; find the equation between p and r. Ans. $p^2 = \dfrac{r^3}{2a}$.

(8) Find the equations between p and r; (1) when $r = a(e^{c+\theta} + e^{c-\theta})$; (2) when $r = a \sec n\theta$.

(9) In the ellipse, if p be the perpendicular from the centre on the tangent, and r = distance of point from the centre, prove that

$$p^2 = \frac{a^2 b^2}{a^2 + b^2 - r^2}.$$

(10) If $r^2 = a^2 \cos 2\theta$ (the Lemniscata): find the equation between p and r: $a^2 p = r^3$.

(11) If $r = a(2\cos\theta \pm 1)$: $p = \dfrac{r^2}{\sqrt{3a^2 \pm 2ar}}$.

(12) If $r^2 = 2a^2 \operatorname{cosec} 2\theta$: $pr = 2a^2$.

(13) If $\theta\sqrt{a^2 - b^2} = a\cos^{-1}\left(\dfrac{\sqrt{a^2 - b^2}}{r}\right)$: $p\sqrt{b^2 + r^2} = ar$.

(14) If $r(\sin^3\theta + \cos^3\theta) = a\cos^2\theta$: the asymptotic subtangent $= -\dfrac{a}{3\sqrt{2}}$.

(15) If $(r - a)\theta^2 = r$: there are both circular and rectilinear asymptotes, and

$$p^2 = \frac{a^2 r^3}{a^2 r + 4(r - a)^3}.$$

(16) If $a\theta = \sqrt{r^2 - a^2} - a\sec^{-1}\left(\dfrac{r}{a}\right)$, (the involute of the circle), $p^2 + a^2 = r^2$.

CHAPTER XIII.

Singular Points in Curves.

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189. If in the equation to a curve expressed by $y = f(x)$, where y is the ordinate, and x the abscissa; some value of x as a makes any of the differential coefficients 0, $\frac{1}{0}$, or $\frac{0}{0}$, the point so determined is called a singular point.

(1°) Let the values of the first differential coefficient be considered.

Since $\frac{dy}{dx}$ represents the tangent of the angle which the tangent makes with the axis of x, if $\frac{dy}{dx} = 0$, the tangent is parallel to the axis of x, and this circumstance generally indicates a maximum or minimum value of the ordinate.

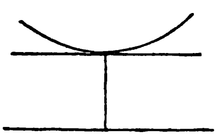
If $\frac{dy}{dx} = \frac{1}{0}$, the tangent is perpendicular to the axis of x.

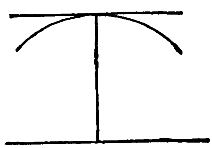
If $y = 0$ when $\frac{dy}{dx} = 0$, then the axis of x is a tangent to the curve at the origin.

If $x = 0$ when $\frac{dy}{dx} = \frac{1}{0}$, then the tangent passes through the origin, and is coincident with the axis of y.

When $\frac{dy}{dx} = \frac{0}{0}$. Many branches may pass through the point, as we shall see in the succeeding pages.

(2°) If $\frac{d^2y}{dx^2}$ have a real value when $\frac{dy}{dx} = 0$, the ordinate is a maximum or minimum, as in the annexed figures.



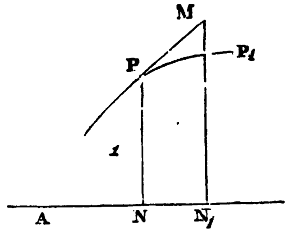


Before we proceed to investigate the values of $\frac{d^2y}{dx^2}$ at these points, we must establish the following proposition.

190. Prop. If the ordinate y be positive, a curve is convex or concave to the axis, according as $\frac{d^2y}{dx^2}$ is positive or negative.

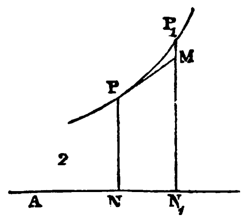
In the annexed figures, let

$\left.\begin{array}{r} AN = x \\ NP = y \\ NN_1 = h \end{array}\right\}$, and $y = f(x)$ be the equation to the curve.



Draw the tangent PM, its equation is

$$y_1 - y = \frac{dy}{dx}(x_1 - x).$$



Now at the point P_1, the equation to the curve becomes

$$N_1P_1 = f(x+h), \text{ or}$$

$$N_1P_1 = y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\frac{h^3}{2.3} + \&c.$$

and for the tangent, putting $x + h$ for x_1, and N_1M for y_1,

$$N_1M = y + \frac{dy}{dx}.h;$$

therefore the deflexion from the tangent, or MP_1

in figure (1) $= N_1M - N_1P_1 = -\frac{d^2y}{dx^2}\frac{h^2}{1.2} - \frac{d^3y}{dx^3}\frac{h^3}{2.3} - \&c.$

in figure (2) $= N_1P_1 - N_1M = +\frac{d^2y}{dx^2}\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\frac{h^3}{2.3} + \&c.$;

and since h^2 is positive, and that h may be taken so small, that the first term of the expansion may be made greater than the sum of all the terms that follow it, the algebraical sign of MP_1 will depend upon that of $\frac{d^2y}{dx^2}$.

Therefore when the curve is concave to the axis, $MP_1 = -\frac{d^2y}{dx^2}\frac{h^2}{2} - \&c.$; and when convex to the axis, it $= +\frac{d^2y}{dx^2}\frac{h^2}{1.2} + \&c.$ Hence, (y being positive,) a curve is convex or concave to the axis, according as $\frac{d^2y}{dx^2}$ is positive

or negative, or generally according as y and $\frac{d^2y}{dx^2}$ have the *same* or *different* signs.

COR. If we suppose PT to be drawn $\perp$ to P_1N_1, $PT=h$,

$$\text{and } \frac{MP'}{PT^2} = \mp \tfrac{1}{2} \cdot \frac{d^2y}{dx^2} \mp \frac{d^3y}{dx^3} \cdot \frac{h}{2.3} + \&c.$$

and if h be constantly diminished, the limit of the ratio of $MP' : PT^2$ will be $= \pm \frac{1}{2} \cdot \frac{d^2y}{dx^2}$.

Hence, ultimately, the deflexion from the tangent,

$$\text{or } MP_1 \propto \frac{d^2y}{dx^2}.$$

191. Sometimes the curve after being convex to the axis suddenly changes its curvature, and becomes concave, the point at which the change takes place is called a point of inflexion, or of *contrary flexure.*

If the tangent at this point be produced, one branch of the curve will be above, and the other below it, consequently on one side of the point in question $\frac{d^2y}{dx^2}$ will be positive, and on the other side, negative. Hence at the point itself $\frac{d^2y}{dx^2}$ must $= 0$, or ∞, for no quantity can change its sign without passing through zero or infinity.

There is not however a point of inflexion corresponding to every value of x, that makes $\frac{d^2y}{dx^2} = 0$, for not only must this equation be satisfied, but $\frac{d^2y}{dx^2}$ must change its sign after having passed through the point under consideration.

Also if the same value of x that makes $\frac{d^2y}{dx^2} = 0$, also makes $\frac{d^3y}{dx^3} = 0$, there may not be a point of contrary flexure.

For since $\frac{d^2y}{dx^2}$ is a function of x, write $x+h$ and $x-h$ for x, and then $\frac{d^2y}{dx^2}$ becomes, on these two suppositions, either

$$f(x+h) = \frac{d^2y}{dx^2} + \frac{d^3y}{dx^3}h + \frac{d^4y}{dx^4}\frac{h^2}{2} + \&c.;$$

$$\text{or } f(x-h)=\frac{d^2y}{dx^2}-\frac{d^3y}{dx^3}h+\frac{d^4y}{dx^4}\frac{h^2}{2}-\&\text{c.}$$

But at a point of inflexion $\frac{d^2y}{dx^2}=0$; $\therefore$ the deflexions from the tangent at points $x+h$ and $x-h$ are respectively proportional to

$$+\frac{d^3y}{dx^3}h+\frac{d^4y}{dx^4}\frac{h^2}{2}+\&\text{c.}, \quad \text{and} \quad -\frac{d^3y}{dx^3}h+\frac{d^4y}{dx^4}\frac{h^2}{2}-\&\text{c.},$$

which have contrary signs if $\frac{d^3y}{dx^3}$ do not $=0$; but if $\frac{d^3y}{dx^3}=0$, and $\frac{d^4y}{dx^4}$ does not vanish, the deflexions before and after the point will have the same algebraical sign, and the branches are both concave, or both convex, to the axis.

And hence in general there may be a point of contrary flexure, when the first differential coefficient which does not vanish is of an odd order.

And, to find whether a curve has a point of inflexion, put $\frac{d^2y}{dx^2}=0$, or $\frac{1}{0}$, and if a be one of the values of x so determined, substitute $a+h$, and $a-h$ for x in the expression for $\frac{d^2y}{dx^2}$. Then if $\frac{d^2y}{dx^2}$ be affected with different signs, $x=a$ gives a point of contrary flexure.

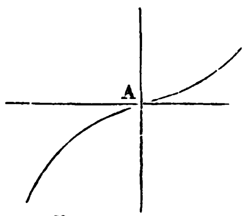
Ex. 1. The cubical parabola

$$a^2y=x^3.$$

$y=\frac{x^3}{a^2}$; and if $x=0$, $y=0$,

$$\frac{dy}{dx}=\frac{3x^2}{a^2},$$

$$\frac{d^2y}{dx^2}=\frac{6x}{a^2}.$$



If x be positive or negative, y and $\frac{d^2y}{dx^2}$ are positive or negative; the curve is therefore always convex to the axis.

If $x=0$, $\frac{d^2y}{dx^2}=0$.

If $x=h$, $\frac{d^2y}{dx^2}=\frac{6h}{a^2}$ is positive.

$$\text{If } x = -h,\ \frac{d^2y}{dx^2} = -\frac{6h}{a^2} \text{ is negative.}$$

The origin is therefore a point of contrary flexure; also, since $x = 0$, makes $\frac{dy}{dx} = 0$ and $y = 0$; the axis of x is a tangent to the curve.

Ex. 2. The Witch. $y = \frac{2a}{x}\sqrt{2ax - x^2}$,

$$\frac{dy}{dx} = \frac{-2a^2}{x\sqrt{2ax - x^2}},$$

$$\frac{d^2y}{dx^2} = 2a^2\ \frac{(3a - 2x)}{x\,.\,(2ax - x^2)^{\frac{3}{2}}},$$

which $= 0$ if $x = \frac{3a}{2}$, and changes its algebraical sign, when $\frac{3a}{2} + h$ and $\frac{3a}{2} - h$ are successively put for x.

There are therefore two points of contrary flexure when

$$x = \frac{3a}{2}, \quad \text{and } y = \pm\frac{2a}{\sqrt{3}}.$$

Ex. 3. Find the point of contrary flexure in the trochoid.

$$y = a(1 - e\cos\theta); \quad x = a(\theta - e\sin\theta);$$

$$\therefore \frac{dy}{d\theta} = ae\sin\theta; \quad \frac{dx}{d\theta} = a(1 - e\cos\theta);$$

$$\therefore \frac{dy}{dx} = \frac{e\sin\theta}{1 - e\cos\theta},$$

$$\frac{d^2y}{dx^2} = \frac{e\cos\theta(1 - e\cos\theta) - e^2\sin^2\theta}{(1 - e\cos\theta)^2}\cdot\frac{d\theta}{dx}$$

$$= \frac{e\cos\theta - e^2}{(1 - e\cos\theta)^2} \times \frac{1}{a(1 - e\cos\theta)}$$

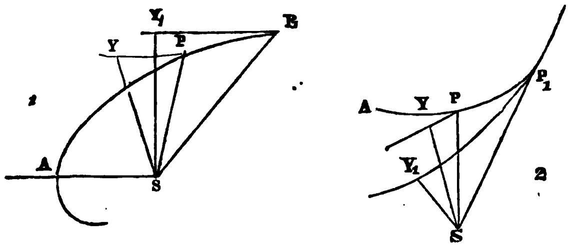
$$= \frac{e(\cos\theta - e)}{a(1 - e\cos\theta)^3} = 0, \text{ if } \cos\theta = e;$$

and $\cos(\theta + h)$ is $< e$, and $\cos(\theta - h) > e$;

$\therefore$ $\cos\theta = e$ gives a point of contrary flexure,

$$\text{and } y = a(1 - e^2) = a\left(1 - \frac{b^2}{a^2}\right) = \frac{a^2 - b^2}{a}.$$

192. Points of contrary flexure in spirals.



Let there be two spirals, one concave and the other convex to the pole. Take two points P and P_1 in each near to each other, and draw SY and $SY_1 \perp$ to the tangents at P and P_1.

Let $SY = p$, $SP = r$, and $SP_1 = r + h$, and $p = f(r)$; therefore if Δ be the difference between SY_1 and SY, we have in figure (1), where the curve is concave to the pole,

$$\Delta = f(r+h) - f(r) = +\frac{dp}{dr} \cdot h + \frac{d^2p}{dr^2}\,\frac{h^2}{1\,.\,2} + \&c.;$$

and in figure (2), where the spiral is convex to S,

$$\Delta = f(r) - f(r+h) = -\frac{dp}{dr} \cdot h - \frac{d^2p}{dr^2}\,\frac{h^2}{1\,.\,2} - \&c.;$$

and as h may be taken so small that $\frac{dp}{dr}\,h$ may be greater than all the terms that follow, we see that the spiral is concave or convex to S, according as $\frac{dp}{dr}$ is positive or negative. Hence at a point of contrary flexure $\frac{dp}{dr} = 0$, and changes its sign immediately before and after the point.

Cor. Since $\frac{1}{p^2} = u^2 + \frac{du^2}{d\theta^2}$, it follows that at a point of contrary flexure $u + \frac{d^2u}{d\theta^2} = 0$.

Example.

Let $r = a\theta^n$, find the point of contrary flexure.

Here $u = \frac{1}{r} = \frac{1}{a}\,\theta^{-n}$;

Note. P_1Y_1 in the Figure should be a straight line.

$$\therefore \frac{du}{d\theta} = -\frac{n}{a}\theta^{-n-1}; \quad \frac{d^2u}{d\theta^2} = \frac{n(n+1)}{a}\theta^{-n-2};$$

$$\therefore \frac{1}{a}\theta^{-n} + \frac{n(n+1)}{a}\theta^{-n-2} = 0,$$

$$\theta^2 + n(n+1) = 0.$$

Hence θ will be impossible, unless $n(n+1)$ is a negative member.

$$\text{Let } n^2 + n = -p; \quad \therefore n + \tfrac{1}{2} = \sqrt{\tfrac{1}{4} - p};$$

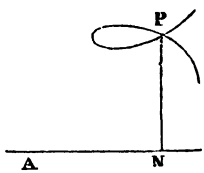
$$\therefore n = -\tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} - p}; \quad \therefore p \text{ must never exceed } \tfrac{1}{4}.$$

If $p = \frac{1}{4}$, $n = -\frac{1}{2}$, and $r = \dfrac{a}{\sqrt{\theta}}$, or $\theta = \dfrac{a^2}{r^2}$ the equation to the lituus.

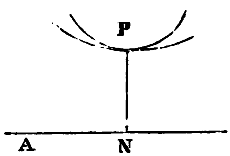
Multiple Points.

193. When two or more branches of a curve pass through a point, it is called a multiple point; and a double, triple, or quadruple point, according as two, three, or four branches pass through it.

If the branches intersect, as in figure (1), which represents a double point, there will be at P two tangents, inclined at different angles to the axis, and thus $\dfrac{dy}{dx}$ will have two values corresponding to one of x or y.



Should however the branches pass through P, as in fig. 2 and touch each other, and the contact be only of the first order, there will be but one value of $\dfrac{dy}{dx}$; but as there are two deflexions from the tangent, there will be two values of $\dfrac{d^2y}{dx^2}$.



194. Problem. If $u = f(x, y) = 0$ be the equation to a curve, cleared of surds, and there be a point where two or more branches intersect, $\dfrac{dy}{dx} = \dfrac{0}{0}$ at that point.

Differentiate the equation, the result will be of the form

$$M \cdot \frac{dy}{dx} + N = 0, \text{ where } M = \frac{du}{dy} \text{ and } N = \frac{du}{dx}.$$

Then since two branches intersect, $\frac{dy}{dx}$ will have two values, but M and N will be the same for both. Let α and β be the two values of $\frac{dy}{dx}$;

$$\therefore M\alpha + N = 0, \text{ and } M\beta + N = 0;$$

$$\therefore M(\alpha - \beta) = 0, \text{ and } \alpha - \beta \text{ does not} = 0;$$

$$\therefore M = 0; \text{ and } \therefore N = 0, \text{ and } \frac{dy}{dx} = -\frac{N}{M} = \frac{0}{0}.$$

The value of p or $\frac{dy}{dx}$ may be found, by the method of vanishing fractions.

Cor. Conversely, to find when there is a multiple point.

Let $u = f(xy) = 0$ be the equation to the curve; find $\frac{dy}{dx} = \frac{P}{Q}$: and if the same values of x and y satisfy $u = 0$, $P = 0$, and $Q = 0$, there may be a multiple point; and there will be, if $\frac{dy}{dx}$ have more real values than one.

Ex. 1. Find the species of point at the origin of the curve,

$$ay^2 - x^3 - bx^2 = 0 = u:$$

$$\text{put } p = \frac{dy}{dx}; \text{ then } 2ayp - 3x^2 - 2bx = 0;$$

$$\therefore p = \frac{3x^2 + 2bx}{2ay} = \frac{0}{0}, \text{ if } x = 0 \text{ and } y = 0,$$

therefore there may be a multiple point, $\because$ $u = 0$ also.

Differentiating both numerator and denominator,

$$p = \frac{6x + 2b}{2ap} = \frac{2b}{2ap}, \text{ when } x = 0;$$

$$\therefore p^2 = \frac{b}{a}, \text{ and } p = \pm\sqrt{\frac{b}{a}};$$

$\therefore$ the origin is a double point, and the tangents cut the axis at angles, $\tan^{-1}\sqrt{\frac{b}{a}}$, and $\tan^{-1}\left(-\sqrt{\frac{b}{a}}\right)$.

This example will be useful in shewing another method by which multiple points may be found. Thus, if there be a surd quantity which disappears from the equation $y=f(x)$ by making $x=a$, but which is found in the equation $\frac{dy}{dx}=\phi(x)$, then $\frac{dy}{dx}$ will have two values, while y has but one, and there is a double point. For resuming the last equation, and solving it with respect to y,

$$y=\pm\frac{x}{\sqrt{a}}\sqrt{x+b}\,;\quad \frac{dy}{dx}=\pm\frac{\sqrt{x+b}}{\sqrt{a}}+\frac{x}{2\sqrt{a}\sqrt{x+b}}.$$

Make $x=0$. Then $y=0$, and $\frac{dy}{dx}=\pm\sqrt{\frac{b}{a}}$, as before.

Ex. 2. Find the point at the origin of the Lemniscata.

$$(x^2+y^2)^2=a^2(x^2-y^2).$$

Here $2(x+py).(x^2+y^2)=a^2(x-py)$;

$$\therefore\ p=\frac{a^2x-2x(x^2+y^2)}{a^2y+2y(x^2+y^2)}=\frac{0}{0},\ \text{if } x \text{ and } y=0,$$

$$=\frac{a^2-2(x^2+y^2)-2x(2x+2yp)}{a^2p+2p(x^2+y^2)+2y(x+2yp)}$$

$$=\frac{a^2}{a^2p},\ \text{if } x=0, \text{ and } y=0\,;$$

$$\therefore\ p^2=1, \text{ and } p=\pm 1,$$

$$\text{or } \frac{dy}{dx}=\pm 1\,; \text{ or } \frac{dy}{dx}=\tan 45, \text{ and } \tan 135.$$

Ex. 3. Find the same, when $x^4-ayx^2+by^3=0$.

Here $4x^3-ax^2p-2ayx+3by^2p=0$;

$$\therefore\ p=\frac{2ayx-4x^3}{3by^2-ax^2}=\frac{0}{0},\ \text{when } x \text{ and } y=0,$$

$$=\frac{2apx+2ay-12x^2}{6bpy-2ax}=\frac{0}{0},\ \text{if } x=0,$$

$$=\frac{2ap+2axq+2ap-24x}{6bp^2+6byq-2a},\quad q=\frac{d^2y}{dx^2},$$

$$=\frac{4ap}{6bp^2-2a},\ \text{if } x=0\,;$$

$$\therefore\ 6bp^3-2ap=4ap,\ \text{ or } p.\{bp^2-a\}=0\,;$$

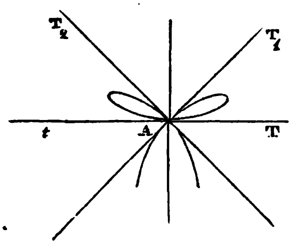
$$\therefore\ p=0, \text{ and } p=\pm\sqrt{\frac{a}{b}},$$

there is a triple point at the origin, and the axis of x is one of the tangents.

The triple point at A is represented in the annexed figure; TAt is the axis of x, AT_1 and AT_2 are the tangents of the angles,

$$\tan^{-1}\sqrt{\frac{a}{b}},$$

$$\text{and } \tan^{-1}\left(-\sqrt{\frac{a}{b}}\right).$$



Ex. 4. Find the same, when $y^3 - 3axy + x^3 = 0$.

$$y^2p - axp - ay + x^2 = 0;$$

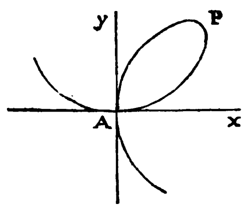
$$\therefore\ p = \frac{ay - x^2}{y^2 - ax} = \frac{0}{0},\ \text{if } x = 0 \text{ and } y = 0;$$

$$\therefore\ p = \frac{ap - 2x}{2yp - a} = \frac{ap}{2yp - a},\ \text{if } x = 0;$$

$$\therefore\ 2yp^2 - ap = ap,$$

$$\text{or } p(yp - a) = 0;$$

$$\therefore\ p = 0, \text{ and } p = \frac{a}{y} = \frac{a}{0} = \infty.$$



The origin is therefore a double point, and the two axes are the tangents. The curve is represented in the figure.

195. If the branches touch, then $\frac{dy}{dx}$ will have but one value, and yet at the same time be of the form $\frac{0}{0}$.

For supposing the contact to be of the n^{th} order between two branches of the curve; then the values of the differential coefficients, as far as the $(n+1)^{\text{th}}$ coefficient, when $x = a$, and $y = b$, will be the same for both branches; but after the n^{th} they will be different.

Let $M\frac{dy}{dx} + N = 0$ be the equation after the first differentiation, the original equation being previously freed from surd quantities.

Then, repeating the differentiation (n) times, we have

$$M \cdot \frac{d^{n+1}y}{dx^{n+1}} + N_1 = 0,$$

M being the same as before, and N_1 being the sum of the differential coefficients below the $(n+1)^{\text{th}}$, together with functions of x or y.

But $\frac{d^{n+1}y}{dx^{n+1}}$ has two values, as α and β, while M and N_1 remain unchanged; $\therefore$ $M.(\alpha-\beta)=0$; $\therefore$ $M=0$.

But $M\frac{dy}{dx}+N=0$; $\therefore$ $N=0$; and $\therefore$ $\frac{dy}{dx}=\frac{0}{0}$.

The analytical character of *double* points of this description is, that when $\frac{dy}{dx}=\frac{0}{0}$ has but one value, $\frac{d^2y}{dx^2}$, which also $=\frac{0}{0}$, has two.

Conjugate or Isolated Points.

196. Conjugate or isolated points are those which have a real existence, and are determined by the equation to the curve; but from which no branches extend.

Hence if $x=a$ and $y=b$ give such a point, then $x=a+h$, and $x=a-h$, will make y, $\frac{dy}{dx}$, and the other differential coefficients, impossible.

Also $\because$ $y_1=f(x+h)=y+\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1.2}+\text{\&c.}$

$$+\frac{d^ny}{dx^n}\frac{h^n}{1\ldots n}+\text{\&c.}$$

is impossible, and that y and h are possible quantities, it is evident that some one of the differential coefficients is impossible, when $x=a$, and $y=b$.

Prop. At a conjugate point, if the equation be freed of surds, $\frac{dy}{dx}=\frac{0}{0}$.

For differentiating the equation, $u=f(xy)=0$, we have

$$M\frac{dy}{dx}+N=0 \ldots\ldots\ldots\ldots(1),$$

and if $\frac{dy}{dx}$ be not impossible, let $\frac{d^ny}{dx^n}$ be impossible;

$\therefore$ continue the differentiation of (1), $(n-1)$ times;

$$\therefore\ M\frac{d^n y}{dx^n}+N_1=0.$$

$$\text{Let } \frac{d^n y}{dx^n}=\alpha+\beta\sqrt{-1};$$

$$\therefore\ M\alpha+N_1+M\beta\sqrt{-1}=0;$$

$$\therefore\ M=0;\quad \therefore \text{ from (1) } N=0;\quad \therefore\ \frac{dy}{dx}=\frac{0}{0},$$

and the values of $\frac{dy}{dx}$ may in general be found, if any, by the method used for finding multiple points.

Ex. 1. $ay^2-x^3+bx^2=0$; $\therefore\ 2ayp-3x^2+2bx=0$;

$$\therefore\ p=\frac{3x^2-2bx}{2ay}=\frac{0}{0}, \text{ if } x=0 \text{ and } \therefore\ y=0$$

$$=\frac{6x-2b}{2ap}=-\frac{b}{ap}, \text{ if } x=0;$$

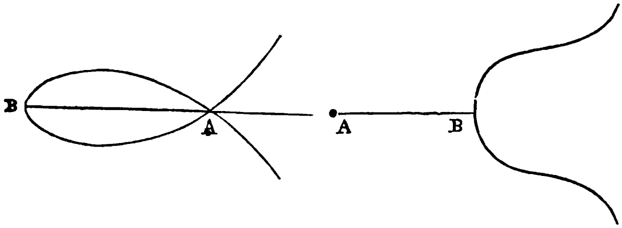
$$\therefore\ p^2=-\frac{b}{a}; \text{ and } \therefore\ p=\sqrt{\frac{-b}{a}}.$$

Now $x=0$ gives $y=0$, while $p=\sqrt{\frac{-b}{a}}$. Also since $y=x\sqrt{\frac{x-b}{a}}$, if $x=0\pm h$, the values of y are impossible, and the origin is therefore a conjugate point. The same result may be obtained by differentiating the equation

$$y=x\sqrt{\frac{x-b}{a}}.$$

For $\frac{dy}{dx}=\sqrt{\frac{x-b}{a}}+\frac{x}{2\sqrt{a}\sqrt{x-b}}=\sqrt{\frac{-b}{a}}$; if $x=0$.

197. The comparison of this example with Ex. 1. in multiple points will serve to explain the origin of conjugate points. In the curve $ay^2-x^3-bx^2=0$; two branches pass through the origin and meet at a point $x=-b$, forming an oval, while in the curve $ay^2-x^3+bx^2=0$, the oval disappears, and no curve exists between the values of $x=0$, and $x=b$; these cases are represented in the annexed figures.



These two examples will shew that points of this kind arise from the vanishing of certain portions of the curve, owing to the change in the value of the constants.

Ex. 2. $y-b=(x-a)^2\sqrt{x-c}$; $a<c$.

If $x=a$, $y=b$; but if $x=a\pm h$, h being very small, so that $a+h$ is $<c$; y is impossible; $\therefore$ $x=a$, $y=b$, determines a conjugate point.

In this example if $x=a$; $\frac{dy}{dx}=0$; but if $q=\frac{d^2y}{dx^2}$; and if the equation be freed of surds,

$$q=\frac{4(x-a)^3+6(x-a)^2(x-c)}{y-b}=\frac{0}{0}; \quad x=a, \quad y=b;$$

$$=\frac{18(x-a)^2+12(x-a)(x-c)}{p}=\frac{0}{0}$$

$$=\frac{48(x-a)+12(x-c)}{q}=\frac{12(a-c)}{q};$$

$$\therefore\ q^2=12(a-c)=-12(c-a);$$

$$\therefore\ q=2\sqrt{3(c-a)}\sqrt{-1};$$

whence we see that $y_1=f(x\pm h)$ is impossible.

198. In general we may remark, if there be a surd which vanishes from the equation $y=f(x)$ if $x=a$, but which becomes impossible in $\frac{d^ny}{dx^n}=\phi(x)$, there will be a conjugate point.

Cusps.

199. When two branches of a curve touch each other at a point through which the branches do not extend, the point is called a Cusp.

The branches have at this point but one tangent, and

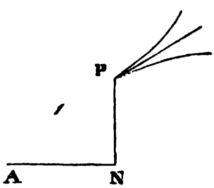
the cusp is said to be of the *first species* when the branches lie on opposite sides of the common tangent, and of the *second species* when they lie upon the same side.

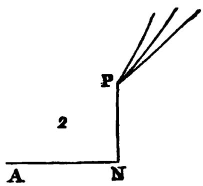
Hence at such a point if $x=a$, $\frac{dy}{dx}$ will have but one value; but if either $a+h$, or $a-h$ be put for x, $\frac{d^2y}{dx^2}$ will have two values.

If the values of $\frac{d^2y}{dx^2}$ be both positive or both negative, the cusp is of the second species; but if one value be positive and the other negative, the cusp is of the first species, for the deflexion of the tangent from the curve is measured by $\frac{d^2y}{dx^2}$.

Since by the definition the branches suddenly stop at the cusp, either $a+h$, or $a-h$, put for x, will make the ordinate and the differential coefficients impossible.

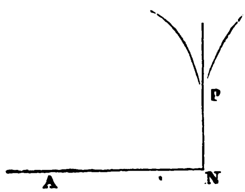
Figures (1) and (2) exhibit cusps of the first and second species.





(1) is called a ceratoid cusp; (2) a ramphoid cusp.

Sometimes the cusp is of the form below,



in which $(a+h)$ and $(a-h)$ put for x give real values for the ordinate.

These are discoverable by observing, that if $x=a$ and $y=b$ give the point P, that $y=b-k$ makes x impossible. Or we may transform the equation to the axis of y making

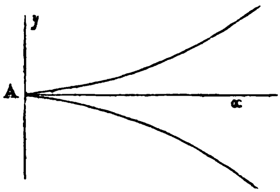
$x = f^{-1}(y)$; and find the values of x, $\frac{dx}{dy}$, and $\frac{d^2x}{dy^2}$ at and near the point where $y = b$. Ex. $(y-2)^3 = (x-1)^2$.

Ex. 1. The semi-cubical parabola.

$$ay^2 = x^3, \quad y = \frac{x^{\frac{3}{2}}}{\sqrt{a}},$$

$$\frac{dy}{dx} = \tfrac{3}{2}\sqrt{\frac{x}{a}},$$

$$\frac{d^2y}{dx^2} = \tfrac{3}{4} \cdot \frac{1}{\sqrt{ax}}.$$

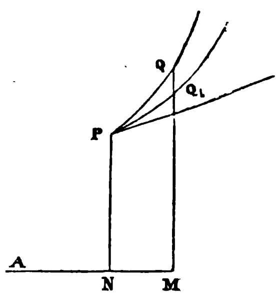


If $x = 0$, y and $\frac{dy}{dx} = 0$, if $x = -h$, they are both impossible. But if $x = h$, y and $\frac{d^2y}{dx^2}$ have two values, one positive and the other negative; the axis of x is therefore a tangent: there are two branches to the curve, one above and the other below the axis of x, and both convex to it, but the curve does not extend through the origin to the negative axis of the abscissas. The origin is a ceratoid cusp.

Ex. 2. Find the point, when $x = a$ in the curve of which the equation is $y = b + cx^2 + (x-a)^{\frac{5}{2}}$;

$$\therefore \frac{dy}{dx} = 2cx + \frac{5}{2}(x-a)^{\frac{3}{2}},$$

$$\text{and } \frac{d^2y}{dx^2} = 2c + \frac{5 \cdot 3}{2 \cdot 2}(x-a)^{\frac{1}{2}}.$$



Let $x = a$; $\therefore y = b + ca^2$,

$$\frac{dy}{dx} = 2ca; \quad \frac{d^2y}{dx^2} = 2c.$$

If $x = a + h$; $y = b + c(a+h)^2 + h^{\frac{5}{2}}$,

$$\frac{dy}{dx} = 2c(a+h) + \tfrac{5}{2} \cdot h^{\frac{3}{2}}; \quad \frac{d^2y}{dx^2} = 2c + \frac{15}{4}h^{\frac{1}{2}};$$

whence in consequence of the index $\frac{1}{2}$, y and $\frac{d^2y}{dx^2}$ have each two values, and those of $\frac{d^2y}{dx^2}$ are both positive, and since $a-h$ put for x makes y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ impossible, the point is a ramphoid cusp.

Take $x=a$, $y=b+ca^2$, and draw a tangent through P inclined to the axis of abscissas at an $\angle = \tan^{-1}2ca$, and then draw two branches above the line through two points Q and Q_1, where

$$MQ = b + c(a+h)^2 + h^{\frac{5}{2}}, \text{ and } MQ_1 = b + c(a+h)^2 - h^{\frac{5}{2}},$$

and the curves will be represented.

200. There are also sometimes to be met with, curves, in which a branch having reached a certain point, is suddenly arrested, and extends only on one side of the point, such a point is called a *stop* point, "*Point d'arrêt.*"

Again, two or more distinct branches sometimes meet at a point, but do not pass through it, nor mutually touch there, each branch having a different tangent. Such points are named shooting points, *points saillants.*

In the former species of point, $y=f(x)$ suddenly changes its value, or from being real, becomes impossible; in the latter $\frac{dy}{dx}$ undergoes a similar change, i. e. from one value to another very different value. These are of rare occurrence; the following are examples.

Ex. 1. The curve where $y = x\log x$, has a stop point at the origin.

$$\text{For } x = +0; \quad y = 0;$$
$$x = -0; \quad y = \infty,$$

Ex. 2. $y = x\tan^{-1}\frac{1}{x}$;

$$\frac{dy}{dx} = \tan^{-1}\frac{1}{x} - \frac{x}{x^2+1}.$$

If $x = +0$; or -0; $y=0$;

$$\frac{dy}{dx} = \frac{\pi}{2}; \quad \text{or } -\frac{\pi}{2}.$$

Hence the origin is a shooting point, the tangents being inclined at angles $\tan^{-1}(1.5708)$, and $\tan^{-1}(-1.5708)$.

201. We shall conclude this Chapter by a few remarks on tracing curves by means of their equations.

(1) If it be possible, let the equation be solved with respect to one of the unknown quantities as y, and let it be put under the form $y = f(x)$.

Then give to x all the possible *positive* values the equation admits of, and so determine the branches above and below the axis of *positive* abscissas.

Next put $(-x)$ for x in the equation $y = f(x)$, and in the equation, thus transformed, again substitute for x all its possible *positive* values, and the branches above and below the axis of *negative* abscissas will be determined.

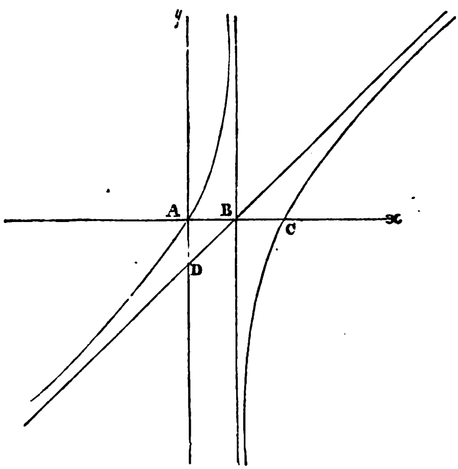
(2) Find whether the curve has asymptotes.

(3) Find whether the branches be concave or convex to the axis, and the nature and situation of the singular points.

These remarks refer to curves having rectangular coordinates, but if the equation be between r and θ, give to θ, values from $\theta = 0$ to $\theta = 2\pi$, and draw the corresponding values of r; the positive values of r at the angles denoted by θ; the negative values, in a directly opposite direction, or separated from the positive by the angle π. Sometimes it may be necessary to take the negative values of θ.

Ex. 1. Let $y = x\left(\dfrac{x-2a}{x-a}\right)$, trace the curve.

A, the origin, *Ax* and *Ay* the two axes.



$$\begin{aligned}
\text{Let } & x=0; \quad \therefore\ y=0,\\
& x<a; \quad \therefore\ y \text{ is positive},\\
& x=a,\ y=\infty,\\
& x>a<2a,\ y \text{ is negative},\\
& x=2a,\ y=0,\\
& x>2a,\ y \text{ is positive},\\
& x=\infty,\ y \text{ is } \infty.
\end{aligned}$$

Again, let $-x$ be put for x;

$$\therefore\ y=-x\frac{x+2a}{x+a} \text{ is always negative.}$$

To draw the asymptote:

$$y=x.\left(\frac{1-\frac{2a}{x}}{1-\frac{a}{x}}\right)=x\left(1-\frac{2a}{x}\right).\{1+\frac{a}{x}+\&c.\};$$

$$\therefore\ y=x\{1-\frac{a}{x}-\frac{2a^2}{x^2}-\&c.\}$$

$$=x-a-\frac{2a}{x}-\&c.;$$

$\therefore\ y=x-a$ is the equation to the asymptote.

Take $\therefore\ AB=a=AD$, and the line BD produced is the asymptote. Also take $AC=2a$. Then since $y=0$, both when $x=0$ and $x=2a$, the curve cuts the axis at A and C.

Between A and B the curve is above the axis; at B the ordinate is infinite; from B to C the curve is below, from C to infinity it is above Ax. Again, since if x be negative, y is negative; the branch on the left of A is entirely below the axis.

$$\text{Also } \frac{dy}{dx}=\frac{x^2-2ax+2a^2}{(x-a)^2}.$$

Let $x=a$; $\therefore\ \frac{dy}{dx}=\infty$; and the infinite ordinate through B is an asymptote; if $x=0$; $\frac{dy}{dx}=2$, or angle at which the curve cuts the axis at A is $\tan^{-1}(2)$; if $x=2a$, $\frac{dy}{dx}$ again $=2$, or $\angle$ at which the curve cuts the axis at C is $=\angle$ at A.

If $(x^2-2ax+2a^2)$ or $(x-a)^2+a^2=0$, x is impossible; hence there is no maximum or minimum ordinate.

Again, $\dfrac{d^2y}{dx^2} = \dfrac{2.(x-a)^2 - 2\{(x-a)^2 + a^2\}}{(x-a)^3} = \dfrac{-2a^2}{(x-a)^3}$;

$$\therefore \frac{d^2y}{dx^2} \text{ is + if } x < a, \text{ and is } - \text{ if } x > a.$$

But $x < a$, y is $+$, and $x > a < 2a$, y is $-$; and $x > 2a$, y is $+$; therefore from A to B, and from B to C the curve is convex, and from C concave to the axis.

If x be $-$, $\dfrac{d^2y}{dx^2} = \dfrac{2a^2}{(x+a)^3}$ is $+$, but y is $-$; therefore the branch from A to the left hand is concave to the axis.

Ex. 2. Let $y^2 = \dfrac{x^3+1}{x-1} = \dfrac{x+1}{x-1}(x^2 - x + 1)$;

$$\therefore y = \pm\sqrt{\frac{x+1}{x-1}} \cdot \sqrt{x^2 - x + 1}.$$

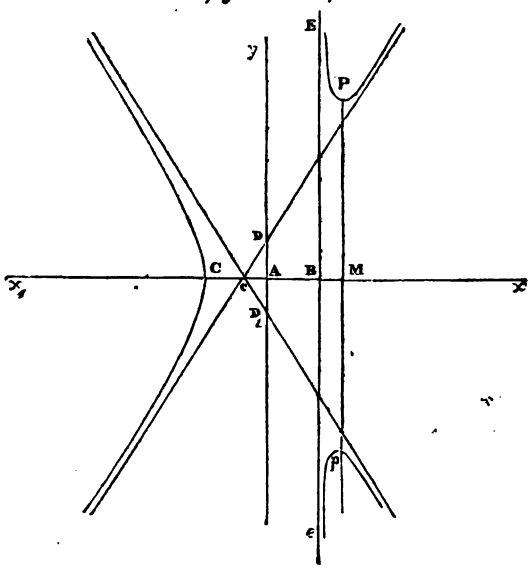
If $x = 0$, $y = \dfrac{1}{\sqrt{-1}}$ is impossible,

$x < 1$, y is impossible,

$x = 1$, y is $\pm\infty$,

$x > 1$, y is possible $\pm$,

$x = \infty$, y is $\infty\ \pm$;



therefore there are two infinite branches extending above and below the axis of positive abscissas.

For x put $-x$; $\therefore\ y=\pm\sqrt{\dfrac{x-1}{x+1}}\,(x^2+x+1)$, which is impossible, if x be <1; and $=0$, if $x=1$.

If $x>1$, and increase to infinity, y is possible $\pm$ and increases to infinity; therefore there are two infinite branches which meet the axis Ax_1 in a point C, if $AC=1$.

To find the asymptote:

$$y=\pm\sqrt{\frac{1+\dfrac{1}{x}}{1-\dfrac{1}{x}}}\cdot x\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}$$

$$=\frac{1+\dfrac{1}{2x}+\&c.}{1-\dfrac{1}{2x}+\&c.}\,x\left(1-\frac{1}{2x}+\frac{1}{2x^2}+\&c.\right)$$

$$=\pm\left\{1+\frac{1}{x}+\&c.\right\}x\left(1-\frac{1}{2x}+\&c.\right)$$

$$=\pm x\left\{1+\frac{1}{2x}+\frac{A}{x^2}+\&c.\right\};$$

$\therefore\ y=\pm(x+\frac{1}{2})$ gives the two asymptotes.

Take $AD=AD_1=\frac{1}{2}$, and $Ac=\frac{1}{2}$. Join cD and cD_1, these lines are the asymptotes, and if through B an infinite ordinate be drawn, two branches of the curve will lie within the angular spaces formed by the intersections of this line with cD and cD_1 produced. These branches of the curve will always lie above the asymptotes, since the ordinate of the asymptote is always less than the ordinate of the curve.

For let y_1 be the ordinate of the asymptote;

$$\therefore\ y^2=\frac{x^3+1}{x-1},\ \text{and}\ y_1^2=x^2+x+\tfrac{1}{4};$$

$$\therefore\ y^2-y_1^2=\frac{x^3+1-(x-1).(x^2+x+\frac{1}{4})}{x-1},$$

$$\text{or}\ \ (y+y_1)(y-y_1)=\frac{3x+1}{4.(x-1)}.$$

But $y+y_1$ is $+$; $\therefore\ y-y_1$ is $+$; or $y>y_1$.

Similarly it may be shewn that the branches which ex-

tend from C, above and below the axis Cx_1, lie between the lines D_1c, and Dc (asymptotes) produced.

To find the values of $\dfrac{dy}{dx}$.

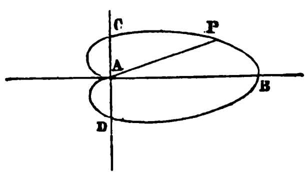
$$2\log y = \log(x^3+1) - \log(x-1);$$

$$\therefore \frac{dy}{dx} = \frac{y}{2}\cdot\left(\frac{3x^2}{x^3+1} - \frac{1}{x-1}\right) = \frac{2x^3-3x^2-1}{2(x-1)^{\frac{3}{2}}\cdot\sqrt{x^3+1}},$$

which is ∞, if $x=1$, or $x=-1$.

Hence the infinite ordinate through B touches the curve at an infinite distance from Ax, or is an asymptote; and the curve at C where $y=0$ cuts the axis at right angles. Also since the numerator $2x^3-3x^2-1$ is $=-2$ when $x=1$, and is $=3$ when $x=2$, there is some value of x between 1 and 2 which will make $\dfrac{dy}{dx}=0$, or y a minimum. Take AM this value, and MP and Mp will be minimum ordinates.

Ex. 3. Let $r = a(1+\cos\theta)$. Trace the curve.



$$\theta = 0; \quad \therefore\ r = a(1+1) = 2a,$$

$$\theta = \frac{\pi}{2}; \quad \therefore\ r = a.$$

Let $\theta = \dfrac{\pi}{2} + \alpha$; $\therefore\ \cos\theta = -\sin\alpha$,

and $r = a(1-\sin\alpha)$, or $r < a$,

$$\alpha = \frac{\pi}{2}, \text{ or } \theta = \pi; \quad \therefore\ r = 0.$$

Let $\theta = (\pi+\alpha)$; $\therefore\ \cos\theta = -\cos\alpha$,

and $r = a(1-\cos\alpha)$, which increases as α increases,

and $r = a$ when $\alpha = 90$.

Let $\theta = \dfrac{3\pi}{2} + \alpha$; $\therefore\ \cos\left(\dfrac{3\pi}{2}+\alpha\right) = +\sin\alpha$;

$\therefore\ r = a(1 + \sin \alpha)$, which increases as α increases, and when $\alpha = \frac{\pi}{2}$, or $\theta = 2\pi$, $r = 2a$.

The curves in the first and fourth quadrants are the same, and also those in the second and third quadrants. If $-\theta$ be put for θ, $\because\ \cos(-\theta) = \cos\theta$: precisely the same curve will be produced. Take $AB = 2a$, $AC = AD = a$, and the points at which it cuts the axes are determined.—This curve is the *Cardioid*.

Examples.

(1) If $y = ax + bx^2 + cx^3$; there is a point of inflexion if $x = -\frac{b}{3c}$.

(2) If $y = x\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}$, trace the curve, find its greatest ordinates, and the angles at which it cuts the axis of x: $x = a\sqrt{\sqrt{2} - 1}$; the angles are 45° and 90°.

(3) Trace the curves defined by the three equations

$$y = x\sqrt{\frac{x^2 - a^2}{x^2 + a^2}}; \quad y = x\sqrt{\frac{x^2 + a^2}{x^2 - a^2}}; \quad y = x\sqrt{\frac{a^2 + x^2}{a^2 - x^2}}.$$

In (1) origin is a conjugate point, two rectilinear asymptotes pass through the origin, and two infinite branches meet the axis of x at $\pm a$. In (2) there are also two rectilinear asymptotes perpendicular to the axis of x: and the branches of the curve are included within the asymptotes. In (3) there are only asymptotes perpendicular to the axis, the branches of the curve pass through the origin and do not extend beyond the asymptotes.

(4) If $y = e^{\cos x}$, there are two points of inflexion corresponding to $x = \cos^{-1}\left(\frac{-1 \pm \sqrt{5}}{2}\right)$.

(5) $y^2 = \frac{a^4}{a^2 + x^2}$; trace the curve: there are inflexions if $x = \pm\frac{a}{\sqrt{2}}$: also when $y^2 = \frac{x^4}{a^2 + x^2}$: draw the asymptote and find the point of flexure.

(6) $y = (x-2)\sqrt{\dfrac{x-9}{x}}$; trace the curve: there is a conjugate point if $x = 2$; and y is a minimum if $x = -\dfrac{3}{2}$.

(7) $y^3 = ax^2 - x^3$, a cusp of the first species at the origin, an inflexion if $x = a$; a maximum ordinate if $x = \dfrac{2a}{3}$.

(8) If $y = \dfrac{a^2x}{a^2+x^2}$, there are two points of flexure when $x = 0$, and $x = a\sqrt{3}$; the curve cuts the axis at 45°, and the axis of x is an asymptote to the two infinite branches; there are maximum ordinates when $x = \pm a$.

(9) If $y = \dfrac{ax^2}{a^2+x^2}$, the curve touches the axis of x at the origin, there is an asymptote parallel to x: and two points of contrary flexure: where $x = \pm\dfrac{a}{\sqrt{3}}$.

(10) If $y^3 = x^3 - a^3$, trace the curve; there are two points of inflexion, when $x = 0$, and when $x = a$.

(11) Find the points of contrary flexure, in the companion to the cycloid, when $x = a(1-\cos\theta)$; $y = a\theta$.

Two points when $x = a$; $y = \pm\frac{1}{2}\pi a$.

(12) Find the point of contrary flexure in the lituus.

$r = a\sqrt{2}$; $\theta = \frac{1}{2}$.

(13) The curve defined by $a^2p = r^3$ has a point of inflexion at the origin.

(14) Determine the nature of the points, when $x = a$.

If (1) $(y-b)^3 = (x-a)^5$, (3) $(y-x)^5 = (x-a)^3$,

(2) $(y-b)^2 = (x-a)^5$, (4) $(y-x)^2 = (x-a)^5$,

and the inclination of the tangents at the points to the axis.

(15) If $x^4 - ax^2y - axy^2 + \frac{1}{4}a^2y^2 = 0$, there is a ramphoid cusp.

(16) If $r = \dfrac{a\theta^2}{\theta^2-1}$, there is a point of contrary flexure when $r = \dfrac{3a}{2}$; there are two rectilinear asymptotes and an asymptotic circle, radius $= a$.

(17) If $x^4 - a^2x^2 + a^3y = 0$, there are two points of contrary flexure.

(18) Trace $x^2.(x+y) = a^2.(x-y)$ and draw the asymptote.

(19) Trace the curve $r = a(2\cos\theta \pm 1)$. The Trisectrix. Like the Cardioid, with the addition of an interior oval.

(20) Trace the curves $y = \sin x$, $y = \tan x$, $y = \sec x$. The circumference is supposed to coincide with the axis of x, and the ordinate is the sine, tangent, or secant. The resulting locus is called, the curve of sines, tangents, or secants.

(21) Trace the curves $r^2 = a^2 \sin 2\theta$, and $r = a \sin 2\theta$. In the former an oval in the 1st and 3rd quadrants: in the latter an oval in every quadrant.

(22) If $r = \pm a \sin 3\theta$, there are six ovals.

(23) Trace $y^3 - 3axy + x^3 = 0$. Make $x = r\cos\theta$, $y = r\sin\theta$: an oval in the 1st quadrant, an asymptote cutting the axis of x at 135^0, two infinite branches in the 2nd and 4th quadrants.

(24) Trace $y^2a = x(a \pm x)^2$. The exterior branch is parabolic: the interior has an oval between $x = 0$ and $x = a$.

(25) Trace the curves, (1) $r = a\tan\theta$; (2) $r = 2a\tan\dfrac{\theta}{2}$; and find the position of the asymptotes. Asymptotic subtangents are a and $4a$; (1) is included between vertical, (2) between horizontal asymptotes.

(26) Trace the curve $r = a\theta$, taking both positive and negative values of θ.

(27) Find the asymptotes, least ordinates, and points of flexure of $y^2x = (x+a)(x-b)^2$.

(28) $(x^2 + y^2)^2 = 4a^2xy$; a double point at the origin, $p = 0$ and $= \infty$.

(29) $(x^2 + y^2)^3 = 4a^2x^2y^2$; a quadruple point at the origin.

(30) $y^4 + 2axy^2 - ax^3 = 0$; a triple point at the origin, $p = \pm\dfrac{1}{\sqrt{2}}$, and $= \infty$.

(31) Given the base of a triangle and the exterior angle equal to three times the interior and opposite angle at the base, find the equation to the curve, which is the locus of the vertex, draw its asymptote, and find its maximum ordinate.

(32) In the diameter AB of a circle, take a point C; draw a chord AP, and an ordinate PN, and CQ parallel to AP, meeting NP in Q: trace the curve which is the locus of Q.

(33) A rod PQ passes through a fixed point A, find the equation to the curve described by P, when Q moves in the circumference of a circle of given radius, and trace the curve.

(34) Two points start from the opposite extremities of the diameter of a circle in the same direction, the velocities are uniform, and in the ratio of 2 : 1; find the locus of the bisection of the chord which joins the position of the points, and its polar subtangent.

CHAPTER XIV.

Curvature and Osculating Curves.

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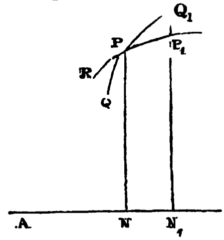
202. WHEN two curves, as QPQ_1, RPP_1, cut each other in the manner represented in the figure, the values of y and x are the same for both curves at the point of intersection; i. e. if $y=f(x)$ be the equation to the curve RPP_1, and $y=\phi(x)$ the equation to QPQ_1, and $AN=a$, and $NP=b$; the values a and b put for x and y will make the equations $b=f(a)$ and $b=\phi(a)$ true equations, and $\therefore f(a)=\phi(a)$.



203. But if for x, $a+h$, be written, (or as we shall put it, $x+h$,) the values of the ordinates of the two curves no longer become equal, and their difference, which is represented in the figure by P_1Q_1, is equal to the difference between $f(x+h)$ and $\phi(x+h)$, and will therefore be some function of h, and its value will depend upon the relations existing between the differential coefficients of $f(x)$ and $\phi(x)$.

For, let $y_1=N_1P_1$, $y_2=N_1Q_1$, $z=f(x)$, and $v=\phi(x)$;

$$\therefore\ y_1=y+\frac{dz}{dx}h+\frac{d^2z}{dx^2}\frac{h^2}{1\,.\,2}+\frac{d^3z}{dx^3}\frac{h^3}{2\,.\,3}+\text{\&c.}$$

$$\text{and } y_2=y+\frac{dv}{dx}h+\frac{d^2v}{dx^2}\frac{h^2}{1\,.\,2}+\frac{d^3v}{dx^3}\frac{h^3}{2\,.\,3}+\text{\&c.};$$

$$\therefore\ P_1Q_1=\left(\frac{dv}{dx}-\frac{dz}{dx}\right)h+\left(\frac{d^2v}{dx^2}-\frac{d^2z}{dx^2}\right)\frac{h^2}{1\,.\,2}+\text{\&c.};$$

or putting A_1, A_2, A_3, &c. A_n for the coefficients of h, h^2, h^3, &c., the distance Δ between the curves, or the difference between the ordinates, is represented by a series with ascending powers of h, so that

$$\Delta=A_1h+A_2h^2+A_3h^3+A_4h^4+\text{\&c.}+A_nh^n+\text{\&c.}$$

204. First, let $A_1=0$; $\therefore \dfrac{dv}{dx}=\dfrac{dz}{dx}$, or the first differential coefficients are equal. But $\dfrac{dv}{dx}$ and $\dfrac{dz}{dx}$ represent

the trigonometrical tangents of the angles which the tangents of the two curves at the point P make with the axis of x. Hence at such a point the ordinates are equal, and the tangents are coincident. This is called a contact of the *first* order.

205. Let not only $A_1 = 0$, but $A_2 = 0$, then

$$f(x) = \phi(x), \quad \frac{df(x)}{dx} = \frac{d\phi(x)}{dx}, \quad \text{and} \quad \frac{d^2 f(x)}{dx^2} = \frac{d^2\phi(x)}{dx^2}.$$

This is called a contact of the *second* order.

And in general the curves are said to have a contact of the n^{th} order when the first power of h, in the expression for Δ is h^{n+1}; i. e. when all the differential coefficients as far as the $(n+1)^{\text{th}}$ are respectively equal in both series.

206. To find the degree of contact which a proposed curve of given species has with a given curve of known dimensions.

Let $y = f(x)$ be the equation to the given curve, and $y_1 = \phi(x_1)$ the equation to the proposed curve, which is supposed to contain n arbitrary constants.

Then, to determine these n constants, we must have the n equations,

$$y = y_1, \quad \frac{dy}{dx} = \frac{dy_1}{dx_1}, \quad \frac{d^2y}{dx^2} = \frac{d^2y_1}{dx_1^2}; \quad \text{and} \quad \frac{d^{n-1}y}{dx^{n-1}} = \frac{d^{n-1}y_1}{dx_1^{n-1}};$$

or the contact must be of the $(n-1)^{\text{th}}$ order.

Thus, let it be required to find the degree of contact which a straight line may have with a given curve; we observe that the equation to the line is $y_1 = ax_1 + b$, and contains two constants a, b, or the contact may be of the first order.

And to determine the straight line which has a contact of the first order with a curve, $y = f(x)$.

Here $\dfrac{dy}{dx} = \dfrac{dy_1}{dx_1} = a$; $y = y_1$, and $x = x_1$;

$$\therefore\ y = ax + b, \quad \text{or } b = y - ax = y - \frac{dy}{dx}x;$$

therefore substituting for a and b, $y_1 = \dfrac{dy}{dx}x_1 + y - x\dfrac{dy}{dx}$;

or $y_1 - y = \dfrac{dy}{dx}(x_1 - x)$ the equation to the tangent; or the tangent has a contact of the first order with the curve which it touches.

COR. Hence no straight line can be drawn nearer than the tangent to the curve.

207. In the circle of which the equation is

$$R^2 = (x_1 - \alpha)^2 + (y_1 - \beta)^2$$

there are three arbitrary constants, the radius R and the co-ordinates of the centre α and β. The circle therefore may have a contact of the second order, and the constants may be determined by means of the equations

$$y = y_1, \quad \frac{dy}{dx} = \frac{dy_1}{dx_1}, \text{ and } \frac{d^2y}{dx^2} = \frac{d^2y_1}{dx_1^2}.$$

The circle so found is called the *circle of curvature*, and its radius the radius of curvature of any point in a given curve.

For since the curvature in the same circle is uniform, while it varies inversely as the radius in different circles, and that curves are *geometrically* said to have the same curvature, when at a common point, they have the same tangent, and ultimately the same deflexion from the tangent, which conditions are fulfilled by the circle that has a contact of the second order; this circle is assumed to be the proper measure of curvature, and curves are said to have the same or different curvature, according as the radii of these circles are the same or different, and the curvature in general

$$\propto \frac{1}{\text{radius of curvature}}.$$

The circle of curvature is also called the osculating circle.

208. To find the radius of curvature, and co-ordinates of the centre of the osculating circle to any proposed curve.

Let $y = f(x)$ be the equation to the given curve,

$R^2 = (x_1 - \alpha)^2 + (y_1 - \beta)^2$ the equation to the circle;

$\therefore$ differentiating twice, we have

$$\therefore \ 0 = (x_1 - \alpha) + (y_1 - \beta) \cdot \frac{dy^1}{dx_1} \ldots\ldots\ldots\ldots(1),$$

$$\text{and } \ 0 = 1 + \frac{dy_1^2}{dx_1^2} + (y_1 - \beta) \cdot \frac{d^2y_1}{dx_1^2} \ldots\ldots\ldots\ldots(2).$$

$$\text{But } \ y = y_1, \ \ x = x_1, \ \ \frac{dy}{dx} = \frac{dy_1}{dx_1}, \text{ and } \frac{d^2y}{dx^2} = \frac{d^2y_1}{dx_1^2};$$

$\therefore$ changing x_1 into x, and y_1 into y;

$$\therefore\ R^2 = (x-\alpha)^2 + (y-\beta)^2$$

$$= (y-\beta)^2 . \left\{1 + \frac{dy^2}{dx^2}\right\} \text{ from} \ldots\ldots(1)$$

$$= \frac{\left(1+\frac{dy^2}{dx^2}\right)^2}{\left(\frac{d^2y}{dx^2}\right)^2} . \left(1+\frac{dy^2}{dx^2}\right) \ldots\ldots\ldots (2)$$

$$= \frac{\left(1+\frac{dy^2}{dx^2}\right)^3}{\left(\frac{d^2y}{dx^2}\right)^2};$$

$$\therefore\ R = \pm \frac{\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \pm \frac{(1+p^2)^{\frac{3}{2}}}{q},$$

where $p = \frac{dy}{dx}$ and $q = \frac{d^2y}{dx^2}$.

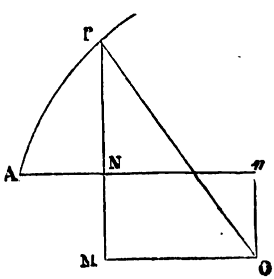
This expression has two signs; but if we call the radius positive, when the curve is concave to the axis, or when q is negative; and if, when the curve is convex, or when q is positive, the radius be reckoned negative, we shall always have $R = \frac{(1+p^2)^{\frac{3}{2}}}{-q}$.

The co-ordinates α and β may be found from the equations

$$y-\beta = -\frac{1+p^2}{q}; \quad x-\alpha = -(y-\beta).\frac{dy}{dx} = \frac{1+p^2}{q}.p;$$

and the circle is completely determined.

209. In the annexed figure, let AP be the given curve, PO the radius of curvature, and O therefore the centre of the osculating circle.



Also let $AN = x$; $NP = y$.

Then $An = \alpha$, $nO = -\beta$;

$$\therefore\ PM = y - \beta = -\frac{(1+p^2)}{q},$$

$$OM = \alpha - x = -\frac{(1+p^2)}{q}.p;$$

PM and OM are respectively called the semi-chords perpendicular and parallel to the axis of x; for if we describe the circle, of which the radius is OP and centre O, PM is half the chord of an arc, since OM is perpendicular to it, and OM is equal to half the chord drawn from P parallel to AN.

210. The point O changes its position with the change in the place of P, and traces out a curve, which is called the *evolute* of the original curve. Hence we may define the evolute to be the locus of the centre of the circle of curvature, and its co-ordinates are α and β.

And since from $y=f(x)$; p and q may be found in terms of y or x,

$$\text{and } \therefore \text{ from } y-\beta=-\frac{1+p^2}{q}; \quad x-\alpha=\frac{1+p^2}{q}.p;$$

and $y=f(x)$; y and x may be eliminated; therefore there will arise an equation between α, β and constant quantities, which will be that of the evolute.

211. Since $x-\alpha+(y-\beta).\dfrac{dy}{dx}=0$;

$$\therefore \beta-y=-\frac{dx}{dy}.(\alpha-x);$$

but this is the equation to the normal of the original curve, drawn from a point, of which the co-ordinates are x, y, and passing through a point whose co-ordinates are α and β. Hence the normal passes through the centre of the circle of curvature, and the radius coincides with the normal.

212. The radius of curvature is a tangent to the evolute.

Resuming the equation $(x-\alpha)+(y-\beta).\dfrac{dy}{dx}=0$.

Differentiate it, considering y, β and α as functions of x;

$$\therefore 1-\frac{d\alpha}{dx}+(y-\beta).\frac{d^2y}{dx^2}+\frac{dy^2}{dx^2}-\frac{d\beta}{dx}.\frac{dy}{dx}=0.$$

$$\text{But } 1+\frac{dy^2}{dx^2}+(y-\beta).\frac{d^2y}{dx^2}=0;$$

$$\therefore -\frac{d\alpha}{dx}-\frac{d\beta}{dx}.\frac{dy}{dx}=0; \quad \therefore \frac{dy}{dx}=-\frac{\frac{d\alpha}{dx}}{\frac{d\beta}{dx}}=-\frac{d\alpha}{d\beta};$$

$$\therefore (x-\alpha)-(y-\beta).\frac{d\alpha}{d\beta}=0; \quad \text{or } (\beta-y)=\frac{d\beta}{d\alpha}.(\alpha-x),$$

which is the equation to a tangent drawn to a point, (β, α), and passing through a point, (y, x).

But $(\beta - y) = \frac{d\beta}{d\alpha}(\alpha - x)$ is identical with

$$(y - \beta) = -\frac{dx}{dy}(x - \alpha),$$

or with the equation to the normal of the original curve.

Hence the normal to the curve, i.e. the radius of curvature, is the tangent to the evolute.

213. To find the length (s) of the evolute.

Since $R^2 = (x - \alpha)^2 + (y - \beta)^2$.

Differentiate, considering R, y, x, β as functions of α,

$$\therefore\ R\frac{dR}{d\alpha} = (x - \alpha)\frac{dx}{d\alpha} + (y - \beta)\frac{dy}{d\alpha} - \left\{x - \alpha + (y - \beta)\frac{d\beta}{d\alpha}\right\}$$

$$= -\left\{x - \alpha + (y - \beta)\cdot\frac{d\beta}{d\alpha}\right\}$$

$$= -(x - \alpha)\left\{1 + \frac{d\beta^2}{d\alpha^2}\right\}.$$

But $R^2 = (x - \alpha)^2\left\{1 + \left(\frac{y - \beta}{x - \alpha}\right)^2\right\}$

$$= (x - \alpha)^2\left\{1 + \frac{d\beta^2}{d\alpha^2}\right\}\ \ldots\ldots\ldots(1).$$

And $R^2\frac{dR^2}{d\alpha^2} = (x - \alpha)^2\cdot\left(1 + \frac{d\beta^2}{d\alpha^2}\right)^2\ldots\ldots(2).$

Divide (2) by (1);

$$\therefore\ \frac{dR^2}{d\alpha^2} = 1 + \frac{d\beta^2}{d\alpha^2} = \frac{ds^2}{d\alpha^2};$$

$\therefore\ R = s + c$, c being some constant length.

Hence, if the equation to the curve be algebraical, R may be found in finite terms, and the length of the evolute found; or the evolutes of algebraic curves are rectifiable.

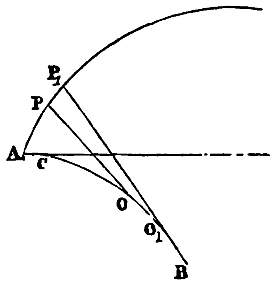
COR. Let s_1, s_2, be two arcs of the evolute, from its commencement to the points where the radii are R_1 and R_2;

$$\therefore\ R_1 - s_1 = c;\quad R_2 - s_2 = c;\quad \therefore\ R_2 - R_1 = s_2 - s_1:$$

$$\text{let } s_2 - s_1 = a;\quad \therefore\ a = R_2 - R_1;$$

or the difference between two radii of curvature equals the length of the arc of the evolute intercepted by them.

214. From this property the curve has derived the name of evolute.



For if we take a string of constant length, one end of which is fastened at B, and the remainder is made to coincide with the curve COO_1B, then if the string be unwrapped or evolved from COO_1B, it will describe the curve APP_1.

COB is called the evolute, and
APP_1 .... involute.

From this construction it is obvious,

(1) That the arc OO_1 is equal to $P_1O_1 - PO$.

(2) That O is the centre of a circle of which the radius OP is the radius of curvature to the point P.

(3) That PO is a tangent to the evolute.

(4) That PO is a normal to the involute.

215. Another, geometrical, method of finding the radius of curvature and the co-ordinates of the centre of the osculating circle is to assume that centre to be the limit of the intersections of two consecutive normals.

For $(y-\beta) = -\dfrac{dx}{dy}\cdot(x-\alpha)$ is the equation to the normal,

$$\text{or } (x-\alpha)+(y-\beta)\cdot\frac{dy}{dx}=0.$$

Now at the point of intersection, α and β remain the same for the two normals, while x, y and $\dfrac{dy}{dx}$ vary, since at a consecutive point, x and y become $x+dx$ and $y+dy$; therefore differentiating, considering α and β as constant,

$$1+\frac{dy^2}{dx^2}+(y-\beta)\cdot\frac{d^2y}{dx^2}=0.$$

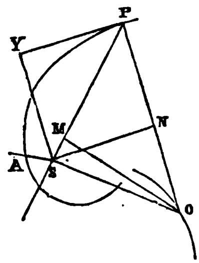
The same equation that has been before obtained to find the co-ordinate β of the centre, and α is then known from

$$x-\alpha=-(y-\beta)\frac{dy}{dx}.$$

And R may be found from the equation

$$R^2 = (x-\alpha)^2 + (y-\beta)^2.$$

216. Hence to find the radius of curvature in spirals.



AP the spiral, S the pole. PO a normal, and O the point of ultimate intersection of two consecutive normals. O is the centre of the circle of curvature.

$$\left.\begin{array}{l} SP = r,\ PO = R \\ SY = p,\ SO = r_1 \end{array}\right\},\quad SN \perp \text{ on } PO = p_1.$$

Now $SO^2 = SP^2 + PO^2 - 2PO.PN$, or $r_1^2 = r^2 + R^2 - 2R.p$; for $PN = SY$.

Then since SO and OP remain constant, while SP and SY vary, and since $p = f(r)$;

$$\therefore\ 0 = r - R\frac{dp}{dr};\quad \therefore\ R = r.\frac{dr}{dp}.$$

If OM be drawn $\perp$ to PS, or PS produced, then $PM = \frac{1}{2}$ the chord of curvature through S,

$$\text{and } PM = PO \times \frac{SY}{SP} = r.\frac{dr}{dp}.\frac{p}{r} = p.\frac{dr}{dp}.$$

217. Evolutes to spirals.

The point O will trace out the evolute, and PO is a tangent to it, and SN is perpendicular to PO, we must therefore find the relation between SO and SN.

$$\text{Now } r_1^2 = r^2 + R^2 - 2Rp \ldots\ldots\ldots(1),$$

$$\text{and } p_1 = PY = \sqrt{r^2 - p^2} \ldots\ldots\ldots(2),$$

$$\text{and } p = f(r) \ldots\ldots\ldots(3),\ \text{and } R = r.\frac{dr}{dp} \ldots\ldots\ldots(4),$$

between these equations p, r and R may be eliminated, and the resulting equation will involve r_1, p_1, and constant quantities, which will be the equation required.

Ex. Let the spiral be the equiangular.

$$\text{Here } p = mr;\ \therefore\ R = \frac{rdr}{dp} = \frac{r}{m},$$

$$p_1 = \sqrt{r^2 - p^2} = r\sqrt{1 - m^2};$$

$$\therefore\ r_1^2 = r^2 + R^2 - 2Rp = r^2 + \frac{r^2}{m^2} - 2r^2$$

$$= \frac{r^2}{m^2} - r^2 = \frac{r^2}{m^2}(1 - m^2) = \frac{p_1^2}{m^2};$$

$$\therefore\ p_1 = mr_1,$$

or the evolute is a spiral similar to the original, and described round the same pole S.

218. If two curves intersect, the distance Δ between them, measured along the ordinate is, (when x becomes $x+h$)

$$\Delta = A_1h + A_2h^2 + A_3h^3 + A_4h^4 + A_5h^5 + \text{\&c.}$$

If we put $(-h)$ for h, the distance Δ_1 between them at a point where x becomes $x-h$; is

$$\therefore\ \Delta_1 = -A_1h + A_2h^2 - A_3h^3 + A_4h^4 - A_5h^5 + \text{\&c.}$$

Now since h may be taken so small that any one term shall exceed the sum of all that follow it; we observe *First,* that if $A_1 = 0$, Δ and Δ_1 have the same sign, or that in a contact of the first order, the curves touch, but do not intersect.

Thus the tangent does not cut the curve, unless $A_2 = 0$, or at a point of contrary flexure.

Secondly. If both $A_1 = 0$ and $A_2 = 0$, or the contact be of the second order. Then

$$\Delta = \quad A_3h^3 + A_4h^4 + \text{\&c.}$$
$$\Delta_1 = -A_3h^3 + A_4h^4 - \text{\&c.},$$

which have different signs, and therefore if the osculating curve be below the given curve at a point where the abscissa is $x+h$, it will be above it at a point where x becomes $x-h$. Hence the circle of curvature both cuts and touches the curve.

There is however an exception to this, when the radius of curvature is a maximum or minimum; for then (as we shall see in the next article) $A_3 = 0$, and the expressions for Δ and Δ_1 have the same sign.

For if the contact be of the third order,

$$\Delta = A_4h^4 + A_5h^5 + \text{\&c.} \quad \Delta_1 = A_4h^4 - A_5h^5 + \text{\&c.};$$

that is, Δ and Δ_1 have the same sign, and therefore the osculating curve does not cut the given curve.

Hence, when the contact is of an even order, the osculating curve both touches and cuts the given curve, but when the contact is of an odd order, it merely touches it.

219. Prop. When the radius of curvature is a maximum or minimum, the contact is of the third order, or $A_3 = 0$.

$$R = \frac{(1+p^2)^{\frac{3}{2}}}{-q}: \quad \frac{dR}{dx} = 0: \text{ and let } r = \frac{d^3y}{dx^3}.$$

$$\therefore\ 3\sqrt{1+p^2}\,.\,p - \frac{(1+p^2)^{\frac{3}{2}}}{q^2} r = 0; \quad \text{or } r = \frac{3pq^2}{1+p^2}.$$

But $1 + p^2 + (y-\beta) q = 0$, and if there be a contact of the third order, we must differentiate this equation, and put the co-ordinates of the curve for those of the circle;

$$\therefore\ 2pq + pq + (y-\beta) r = 3pq - \frac{1+p^2}{q} r = 0;$$

$$\therefore\ r = \frac{3pq^2}{1+p^2}.$$

The same as before, and therefore when $A_3 = 0$; for the circle and the curve, or when the contact is of the third order, the radius of curvature is either a maximum or minimum.

220. If $q = 0$; and p is finite, $R = \infty$; this takes place at a point of contrary flexure; for the curve changes from convex to concave; the circle of curvature becomes a straight line, (the tangent), and before and after the point the radius of curvature is measured in opposite directions.

Examples.

(1) Find the radius of curvature and evolute of the common parabola.

$$y^2 = 4ax, \quad \frac{dy}{dx} = \frac{2a}{y}, \quad \frac{d^2y}{dx^2} = -\frac{2a}{y^2}\frac{dy}{dx} = -\frac{4a^2}{y^3},$$

$$1 + p^2 = 1 + \frac{4a^2}{y^2} = \frac{4a^2 + y^2}{y^2} = \frac{4a(a+x)}{y^2};$$

$$R = \frac{(1+p^2)^{\frac{3}{2}}}{-q} = \frac{\{4a\,.(a+x)\}^{\frac{3}{2}}}{4a^2} = \frac{2(a+x)^{\frac{3}{2}}}{\sqrt{a}} = \frac{2}{\sqrt{a}}\,.\,SP^{\frac{3}{2}}.$$

$$(y-\beta) = \frac{1+p^2}{-q} = \frac{4a^2+y^2}{y^2}\,\frac{y^3}{4a^2} = y\,.\,\frac{(a^2+ax)}{a^2} = y + \frac{yx}{a};$$

$$\therefore\ -\beta = \frac{yx}{a} = \frac{y^3}{4a^2};\quad \therefore\ -y = (4a^2\beta)^{\frac{1}{3}},$$

$$x - \alpha = -p\,.\,(y - \beta) = -\frac{2a}{y}\,\frac{y(x+a)}{a} = -2(x+a);$$

$$\therefore\ 3x = \alpha - 2a, \text{ or } x = \left(\frac{\alpha - 2a}{3}\right).$$

$$\text{But } \because\ y^2 = 4ax;\quad \therefore\ (4a^2\beta)^{\frac{2}{3}} = \frac{4a}{3}(\alpha - 2a);$$

$$\therefore\ \beta^2 = \frac{4}{27a}\,.\,(\alpha - 2a)^3 = \frac{4}{27a}x_1^{\,3};\quad \text{if } \alpha - 2a = x_1,$$

the equation to the semi-cubical parabola.

(2) In the Conic Sections, $R \propto (\text{normal})^3$.

$$\text{Normal} = N = y\sqrt{1+p^2};\quad \therefore\ \sqrt{1+p^2} = \frac{N}{y};$$

$$\therefore\ R = \frac{(1+p^2)^{\frac{3}{2}}}{-q} = \frac{N^3}{-y^3 q}.$$

Now if the vertex be the origin, and the axis the axis of x,

$$y^2 = 2mx + nx^2;\quad \therefore\ yp = m + nx;$$

$$q\quad \therefore\ yp + p^2 = n;\quad \therefore\ y^3 q = ny^2 - p^2y^2$$

$$= n(2mx + nx^2) - (m + nx)^2 = -m^2;$$

$$\therefore\ R = \frac{N^3}{m^2}.$$

(3) Find the radius of curvature of the ellipse.

$$y = \frac{b}{a}\sqrt{a^2 - x^2};\quad \therefore\ p = -\frac{b}{a}\,\frac{x}{\sqrt{a^2 - x^2}}$$

$$-q = \frac{ba}{(a^2 - x^2)^{\frac{3}{2}}},$$

$$1 + p^2 = 1 + \frac{b^2x^2}{a^2(a^2 - x^2)} = \frac{a^4 - (a^2 - b^2)x^2}{a^2(a^2 - x^2)} = \frac{a^2 - e^2x^2}{a^2 - x^2};$$

$$\therefore\ R = \frac{(a^2 - e^2x^2)^{\frac{3}{2}}}{ba}.$$

COR. Let R_1 be the radius at the vertex, and R_2 the radius at the extremity of the minor axis;

$$\therefore\ R_1 = \frac{(a^2 - a^2e^2)^{\frac{3}{2}}}{ba} = \frac{b^2}{a};\quad R_2 = \frac{a^3}{ba} = \frac{a^2}{b};$$

the length of the evolute of the elliptic quadrant

$$= R_2 - R_1 = \frac{a^2}{b} - \frac{b^2}{a} = \frac{a^3 - b^3}{ab};$$

If $R =$ a maximum; $-3\sqrt{a^2 - e^2x^2}\,.\,e^2x = 0$;

$\therefore\ x = 0$, and $x^2 = \dfrac{a^2}{e^2}$; but $x = \dfrac{a}{e}$ or $> a$ is impossible,

$$\text{and } \frac{d^2R}{dx^2} = -3e^2\sqrt{a^2 - e^2x^2} + 3e^2x\frac{d}{dx}\sqrt{a^2 - e^2x^2}$$

$$= -3e^2a, \text{ if } x = 0;$$

therefore R is a maximum, when $x = 0$, or $y = \pm b$.

Hence, at the extremities of the minor axis the circle of curvature touches the ellipse.

(4) To find the evolute of the ellipse,

$$y - \beta = \frac{1 + p^2}{-q} = \frac{(a^2 - e^2x^2)\sqrt{a^2 - x^2}}{ba} = \frac{y(a^2 - e^2x^2)}{b^2};$$

$$\therefore\ \beta = -y\,.\left\{\frac{a^2 - e^2x^2}{b^2} - 1\right\} = -\frac{ye^2}{b^2}(a^2 - x^2) = -\frac{y^3e^2\,.\,a^2}{b^4};$$

$$\therefore\ \left(\frac{y}{b}\right)^3 = -\frac{b\beta}{(ae)^2};\quad \therefore\ \left(\frac{y}{b}\right)^2 = \frac{(b\beta)^{\frac{2}{3}}}{(ae)^{\frac{4}{3}}},$$

$$x - a = -y\,.\left(\frac{a^2 - e^2x^2}{b^2}\right)\frac{dy}{dx} = y\,.\left(\frac{a^2 - e^2x^2}{b^2}\right).\frac{b^2x}{a^2y}$$

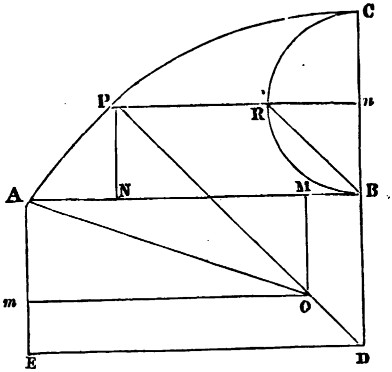
$$= \frac{(a^2 - e^2x^2)}{a^2}x = x - \frac{e^2x^3}{a^2};$$

$$\therefore\ \left(\frac{x}{a}\right)^3 = \frac{-a}{ae^2};\quad \therefore\ \frac{x^2}{a^2} = \frac{(a)^{\frac{2}{3}}}{a^{\frac{2}{3}}e^{\frac{4}{3}}} = \frac{(aa)^{\frac{2}{3}}}{(ae)^{\frac{4}{3}}}.$$

But $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$; $\therefore\ \dfrac{(aa)^{\frac{2}{3}}}{(ae)^{\frac{4}{3}}} + \dfrac{(b\beta)^{\frac{2}{3}}}{(ae)^{\frac{4}{3}}} = 1$;

$$\therefore\ (aa)^{\frac{2}{3}} + (b\beta)^{\frac{2}{3}} = (ae)^{\frac{4}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

(5) **Radius of curvature and evolute of cycloid.**



$$\left.\begin{array}{l} AN = x \\ NP = y \\ CB = 2a \end{array}\right\}; \quad \therefore \frac{dy}{dx} = \frac{\sqrt{2a-y}}{\sqrt{y}}; \quad \therefore 1 + \frac{dy^2}{dx^2} = \frac{2a}{y};$$

$$\therefore \frac{dy}{dx} \cdot \frac{d^2y}{dx^2} = \frac{-a}{y^2}\frac{dy}{dx}; \quad \therefore -\frac{d^2y}{dx^2} = \frac{a}{y^2};$$

$$\therefore R = \left(\frac{2a}{y}\right)^{\frac{3}{2}} \cdot \frac{y^2}{a} = 2\sqrt{2ay} = 2RB.$$

To find the evolute.

$$y - \beta = \frac{2a}{y}\frac{y^2}{a} = 2y; \quad \therefore \beta = -y,$$

$$x - \alpha = -(y-\beta)\frac{dy}{dx} = -2y\frac{\sqrt{2ay-y^2}}{y} = -2\sqrt{2ay-y^2};$$

$$\therefore \alpha = x + 2\sqrt{2ay-y^2};$$

$$\therefore \frac{d\alpha}{dx} = 1 + \frac{2(a-y)}{\sqrt{2ay-y^2}} \cdot \frac{dy}{dx} = 1 + \frac{2(a-y)}{y} = \frac{2a-y}{y},$$

$$\text{and } -\frac{d\beta}{dx} = \frac{dy}{dx} = \sqrt{\frac{2a-y}{y}};$$

$$\therefore -\frac{d\alpha}{d\beta} = \sqrt{\frac{2a-y}{y}} = \sqrt{\frac{2a-(-\beta)}{(-\beta)}}.$$

Take $Am = x_1 = -\beta$, and $mO = \alpha = y_1$;

$$\therefore -\frac{d\alpha}{d\beta} = \frac{dy_1}{dx_1} = \sqrt{\frac{2a-x_1}{x_1}} = \frac{\sqrt{2ax_1-x_1^2}}{x_1}.$$

NOTE—*AOD* in the figure should be a curve.

The equation to a cycloid, of which the vertex is A, and the diameter of the generating circle $=2a$.

(6) Find the chords of curvature drawn through the centre and focus of an ellipse.

Since if $CP=r$, and CY, $\perp$ to tangent $=p$,

$$p^2=\frac{a^2b^2}{a^2+b^2-r^2};$$

$$\therefore\ 2\log p=\log a^2b^2-\log(a^2+b^2-r^2);$$

$$\therefore\ \frac{dp}{pdr}=\frac{r}{a^2+b^2-r^2};$$

$$\therefore\ \text{chord through centre}=\frac{2pdr}{dp}=\frac{2(a^2+b^2-r^2)}{r}=\frac{2CD^2}{CP},$$

$$\text{diameter}=2r\frac{dr}{dp}=\frac{2(a^2+b^2-r^2)}{p}=\frac{2CD^2}{PF}.$$

COR. The diameter $=\dfrac{2(a^2+b^2-r^2)^{\frac{3}{2}}}{ab}=\dfrac{2CD^3}{CA\,.\,CB}$.

(7) To find the chord through the focus.

Here if $SP=r$, $SY=p$; $p^2=\dfrac{b^2r}{2a-r}$;

$$\therefore\ 2\log p=\log b^2+\log r-\log(2a-r);$$

$$\therefore\ \frac{2dp}{pdr}=\frac{1}{r}+\frac{1}{2a-r}=\frac{2a}{r\,.\,(2a-r)};$$

$$\therefore\ \text{chord}=2p\frac{dr}{dp}=\frac{2r(2a-r)}{a}=\frac{2SP\,.\,HP}{AC}=\frac{2CD^2}{AC}.$$

(8) Find the form of the parabola $y=a+bx+cx^2$, which has a contact of the second order, with a given curve at a given point.

Make the given point the origin: then the equation to the parabola becomes $y=bx+cx^2$; and let $y=f(x)$ be the equation to the given curve, from which find p and q.

$$\text{But from above } \frac{dy}{dx}=b+2cx;\ \text{ and } \frac{d^2y}{dx^2}=2c;$$

$$\therefore\ p=b+2cx;\ \text{ and } q=2c.$$

But at the origin $x=0$; $\therefore$ $b=p$, and $c=\dfrac{q}{2}$;

$$\therefore\ y=px+\frac{qx^2}{2}=\frac{q}{2}\cdot\left(x^2+\frac{2p}{q}x+\frac{p^2}{q^2}\right)-\frac{p^2}{2q};$$

$$\therefore \left(y + \frac{p^2}{2q}\right) = \frac{q}{2} \cdot \left(x + \frac{p}{q}\right)^2.$$

The equation to a parabola, of which the axis is perpendicular to the axis of x, the co-ordinates of the vertex $-\frac{p^2}{2q}$, and $-\frac{p}{q}$; and the latus rectum $= \frac{2}{q}$.

Cor. The general equation to the second degree, or

$$y^2 + (ax + b)\,y + cx^2 + ex + f = 0,$$

containing five constants, may have a contact of the fourth order, with a curve. And should there be a point at which $a^2 - 4c = 0$, the osculating curve is a parabola. Immediately before and after this point, a^2 must be greater or less than $4c$; and therefore the osculating parabola is intermediate between an osculating ellipse and hyperbola.

Examples.

(1) In the cubical parabola where $a^2 y = x^3$;

$$R = -\frac{(a^4 + 9x^4)^{\frac{3}{2}}}{6a^4 x}.$$

(2) In the semicubical parabola where $ay^2 = x^3$;

$$R = -\frac{(4a + 9x)^{\frac{3}{2}}}{6a} \sqrt{x}.$$

(3) If $y^2 + x^2 = ax - ay$; an equation to the circle, $R = \frac{a}{\sqrt{2}}$.

(4) The equation to the hyperbola being $y^2 = \frac{b^2}{a^2}(x^2 - a^2)$; $R = \frac{(e^2x^2 - a^2)^{\frac{3}{2}}}{ab}$; and the equation to the evolute is

$$(a\alpha)^{\frac{2}{3}} - (b\beta)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}.$$

(5) In the parabola the chord of curvature through the focus $= 4SP$; and the length of the evolute

$$= \frac{2\,(SP^{\frac{3}{2}} - SA^{\frac{3}{2}})}{\sqrt{SA}}.$$

(6) If $yx = a^2$, $R = -\frac{(x^4 + a^4)^{\frac{3}{2}}}{2a^2x^3}$, and equation to evolute is

$$(\alpha + \beta)^{\frac{2}{3}} - (\alpha - \beta)^{\frac{2}{3}} = (4a)^{\frac{2}{3}}.$$

(7) The equation to the catenary is $2y = a\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$; shew that the radius of curvature is equal, but opposite, to the normal. $R = -\frac{y^2}{a}$.

(8) If $r = a(1 + \cos\theta)$; the radius of curvature $= \frac{2\sqrt{2ar}}{3}$; and chord $= \frac{4r}{3}$.

(9) In the spiral of Archimedes, the radius = the chord of curvature, when $r = \frac{a}{\sqrt{3}}$.

(10) The evolute of the epicycloid, of which the equation is $p^2 = e^2 . \frac{r^2 - a^2}{e^2 - a^2}$, is another epicycloid,

$$p_1^2 = e^2 . \frac{r_1^2 - a_1^2}{e^2 - a^2}; \text{ and } a_1 = \frac{a^2}{e}.$$

(11) Find the chords of curvature drawn through the centre and focus of an hyperbola.

(1°) If $r = CP$; chord $= \frac{2(r^2 - a^2 + b^2)}{r}$.

(2°) If $r = SP$; chord $= \frac{2r(2a + r)}{a}$.

(12) If $y\sqrt{1 + \frac{dx^2}{dy^2}} = c$, be the equation to a curve (the Tractrix); the equation to the evolute is $\frac{d\beta}{d\alpha} = \frac{\sqrt{\beta^2 - c^2}}{c}$ (the Catenary).

(13) In the focal distance SP of a parabola, take $SQ = PN$; find the equation to the locus of Q, and the radius of curvature.

$$SQ = r, \quad SA = a, \quad \angle ASQ = \theta; \quad r = 2a \tan\frac{\theta}{2}.$$

$$R = \frac{(r^4 + 24a^2r^2 + 16a^4)^{\frac{3}{2}}}{32a^3(4a^2 + 3r^2)}.$$

(14) Determine that point in a cubical parabola, where the curvature is the greatest: and the point in the common parabola where it is $\frac{1}{8}$th of the greatest curvature.

$$(1)\ x = \frac{a}{\sqrt[4]{45}}, \qquad (2)\ x = 3a.$$

(15) If $a\theta = \sqrt{r^2 - a^2} - a \sec^{-1}\left(\frac{r}{a}\right)$, (the involute of the circle), shew that $p = \sqrt{r^2 - a^2}$; and find the equation to the evolute.

(16) In the parabola if D be the point where the axis intersects the directrix, and PN, QM be ordinates of corresponding points in the parabola and evolute, prove that $DM = 3DN$.

(17) If λ be the angle which the normal makes with the axis of x in the ellipse, then

$$R = \frac{a(1-e^2)}{(1-e^2\sin^2\lambda)^{\frac{3}{2}}}.$$

(18) In the curve (hypocycloid) of which the equation is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$; the equation to the evolute is

$$(\alpha+\beta)^{\frac{2}{3}} + (\alpha-\beta)^{\frac{2}{3}} = 2a^{\frac{2}{3}}.$$

(19) Let R and R_1 be the radii of curvature of the extremities of two conjugate diameters of an ellipse, then

$$(R^{\frac{2}{3}} + R_1^{\frac{2}{3}})(ab)^{\frac{2}{3}} = a^2 + b^2.$$

(20) When the angle between the radius vector of a spiral and the perpendicular on the tangent is a maximum $R = \frac{r^2}{p}$.

(21) Two normals at two points in a parabola, on opposite sides of the axis, the ordinates being as 1 : 2, intersect in the evolute.

CHAPTER XV.

Envelopes to Curves. Caustics.

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221. WHEN a curve touches a series of curves, all described after a given law, the former is said to be an envelope of the latter; these latter are of a given form, and the problem is to find the touching curve or envelope.

For the better explanation of this application of the Differential Calculus, let us suppose that it was required to find the equation to the curve, touching any number of equal circles, whose centres are in a known curve.

Then if y and x be the co-ordinates of the touching curve, α and β those of the centre of one of the circles

$$(y-\beta)^2+(x-\alpha)^2=r^2.$$

But β and α are the co-ordinates of the known curve;

$$\therefore\ \beta=\phi(\alpha);$$

$$\therefore\ \{y-\phi(\alpha)\}^2+(x-\alpha)^2=r^2\ldots\ldots\ldots(1).$$

Now if we suppose α to receive an indefinitely small increment, the equation (1) will belong to an equal circle, the centre of which is indefinitely near to that denoted by equation (1); and the two circles will intersect at a point of which the co-ordinates are ultimately x and y; and similarly proceeding with a third and other circles, we may conceive the touching curve to be formed by the continual intersections of these circles: and to determine its equation, which must be independent of α, α must be eliminated between the equations $\{y-\phi(\alpha)\}^2+(x-\alpha)^2=r^2$, and the equation which indicates that we have passed from the consideration of one circle to the other, that is, the differential of the equation (1), taken with respect to α.

Hence we may conclude, that if $V=f(xy\alpha)=0$ represent the equation of one of the given curves, the touching curve may be found by eliminating α between the equations

$$V=0,\ \text{and}\ \frac{dV}{d\alpha}=0.$$

222. That $V=0$, and $\dfrac{dV}{d\alpha}=0$, are simultaneous equa-

tions, may be also thus shewn. Resuming the equation to the circle.

Let $\alpha + \delta\alpha$, and $\beta + \delta\beta$ be the values of α and β in the consecutive circle;

$$\therefore\ \{x - (\alpha + \delta\alpha)\}^2 + \{y - (\beta + \delta\beta)\}^2 = r^2;$$

therefore by subtraction,

$$(x - \alpha)^2 - \{x - (\alpha + \delta\alpha)\}^2 + (y - \beta)^2 - \{y - (\beta + \delta\beta)\}^2 = 0,$$

$$\text{or } \delta\alpha\,\{2\,.\,(x - \alpha) - \delta\alpha\} + \delta\beta\,\{2\,.\,(y - \beta) - \delta\beta\} = 0,$$

$$\text{or } 2(x - \alpha) + 2\,.\,(y - \beta)\frac{\delta\beta}{\delta\alpha} - \{\delta\beta\frac{\delta\beta}{\delta\alpha} + \delta\alpha\} = 0.$$

Now make $\delta\alpha = 0$; and $\delta\beta = 0$; then the point of intersection of the two circles becomes a point in the touching curve; $\frac{\delta\beta}{\delta\alpha}$ becomes the differential coefficient of β with respect to α, or $\frac{d\beta}{d\alpha}$, and $2(x - \alpha) + 2(y - \beta)\frac{d\beta}{d\alpha} = 0$, which is the differential coefficient of $(x-\alpha)^2 + (y-\beta)^2 = r^2$ with respect to α, between which two equations α may be eliminated.

Prob. I. Find the curve which shall touch all the straight lines defined by the equation $y = ax + r\sqrt{a^2 + 1}$, r being a perpendicular of constant length from the origin upon the lines.

Differentiating with respect to a; x and y being constant,

$$x + \frac{ra}{\sqrt{a^2 + 1}} = 0; \quad \therefore\ \frac{r}{x} = -\frac{\sqrt{a^2 + 1}}{a};$$

$$\therefore\ \frac{1}{a^2} = \frac{r^2}{x^2} - 1 = \frac{r^2 - x^2}{x^2}; \quad \therefore\ a = \frac{x}{\sqrt{r^2 - x^2}};$$

$$\text{and } \sqrt{a^2 + 1} = -a\frac{r}{x} = -\frac{r}{\sqrt{r^2 - x^2}};$$

$$\therefore\ y = ax + r\sqrt{a^2 + 1}$$

$$= \frac{x^2}{\sqrt{r^2 - x^2}} - \frac{r^2}{\sqrt{r^2 - x^2}} = -\frac{r^2 - x^2}{\sqrt{r^2 - x^2}} = -\sqrt{r^2 - x^2};$$

$$\therefore\ y^2 + x^2 = r^2, \text{ the equation to a circle.}$$

Prob. II. A given straight line slides between two rectangular axes, find the curve to which it is always a tangent.

Let c be the length of the line, a and b the parts of the axes cut off in any given position of the line;

$$\therefore \frac{x}{a} + \frac{y}{b} = 1, \text{ and } a^2 + b^2 = c^2;$$

$$\therefore \frac{x}{a^2} + \frac{y}{b^2}\frac{db}{da} = 0, \text{ and } a + b\frac{db}{da} = 0; \quad \therefore \frac{db}{da} = -\frac{a}{b};$$

$$\therefore \frac{x}{a^2} - \frac{ya}{b^3} = 0; \quad b = a\sqrt[3]{\frac{y}{x}}; \quad \therefore a^2 + b^2 = a^2\left\{\frac{y^{\frac{2}{3}} + x^{\frac{2}{3}}}{x^{\frac{2}{3}}}\right\} = c^2;$$

$$\therefore a = c\frac{x^{\frac{1}{3}}}{\sqrt{y^{\frac{2}{3}} + x^{\frac{2}{3}}}}; \quad b = c\frac{y^{\frac{1}{3}}}{\sqrt{x^{\frac{2}{3}} + y^{\frac{2}{3}}}};$$

$$\therefore \frac{x}{a} + \frac{y}{b} = \frac{\sqrt{y^{\frac{2}{3}} + x^{\frac{2}{3}}}}{c}\{x^{\frac{2}{3}} + y^{\frac{2}{3}}\} = 1;$$

$$\therefore (x^{\frac{2}{3}} + y^{\frac{2}{3}})^{\frac{3}{2}} = c, \text{ and } x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}.$$

Prob. III. If $a^n + b^n = c^n$, the equation to the envelope is

$$x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}} = c^{\frac{n}{n+1}}.$$

Prob. IV. Find the curve which touches all the lines defined by $y = mx + \sqrt{m^2a^2 + b^2}$; a and b being constant.

$$a^2y^2 + b^2x^2 = a^2b^2.$$

Prob. V. Find the curve which touches all the ellipses described round the same centre and with coincident axes, the rectangle of the axes being a constant area (m^2).

$$\text{Here } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1: \text{ and } ab = m^2;$$

$\therefore 2xy = m^2$, the equation to the rectangular hyperbola.

Prob. VI. Find the equation to the curve whose tangent cuts off from the axes two lines the sum of which $= c$.

$$\sqrt{x} + \sqrt{y} = \sqrt{c}.$$

Prob. VII. Find the curve which touches all the curves included under the equation $y = x\tan\theta - \dfrac{x^2}{4h\cos^2\theta}$, θ being supposed variable.

$$x^2 = 4h(h - y).$$

Prob. VIII. Find the curve when $AD^m = c^{m-1} . AT$.

Prob. IX. Find the curve, when the rectangle contained by two lines, drawn perpendicular to the axis of x, one from the origin, the other from a given point in the axis, to meet the tangent, is $= b^2$.

Prob. X. Find the curve whose tangent cuts off from the axes a constant area; the axes being first rectangular, secondly oblique.

Prob. XI. Find the same as in problem 7, when h and θ both vary; but $m^2 = h^2 \sin^2\theta . \cos\theta$; m^2 being a constant area.

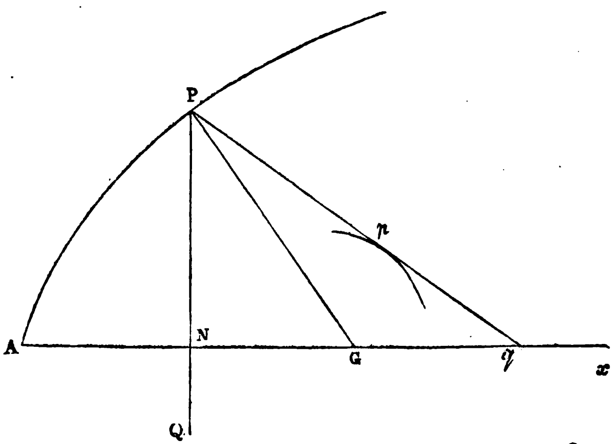
$$xy = \frac{64m^2}{27}.$$

Prob. XII. Two diameters of a circle intersect at right angles, find the locus of the intersections of the chords joining the extremities of the diameters, while the diameters perform a complete revolution. Ans. $x^2 + y^2 = \frac{a^2}{2}$.

Caustics.

223. By the same method as that used in the preceding article, the equations to the curves formed by the intersection of rays reflected by a surface, or refracted through a medium may be found. These curves are called Caustics. Some of them may be practically exhibited by means of a ring of metal, placed on a sheet of paper and held towards the rays of the sun: the curved part of the sugar tongs may be thus used.

224. Prob. Rays of light fall perpendicularly to the axis of x, find the equation to one of the reflected rays, and the equation to the curve of their intersection, or the Caustic.



QP one of the incident rays,

Pq a reflected ray, making with the normal PG,

$$\angle qPG = \angle GPN.$$

$AN = x$, $\theta = \angle NPG$,

$NP = y$, Y and X the co-ordinates of Pq;

$\therefore\ Y - y = m(X - x)$ is the equation to Pq.

But $m = \tan Pqx = \tan(90 + 2\theta) = -\cot 2\theta$

$$= -\left(\frac{1-p^2}{2p}\right), \text{ where } p = \frac{dy}{dx};$$

$$\therefore\ Y - y = -\left(\frac{1-p^2}{2p}\right)(X - x),$$

the equation to the reflected ray.

Now differentiating, Y and X being constant, and $q = \frac{dp}{dx}$;

$$\therefore\ -p = \frac{1-p^2}{2p} + \tfrac{1}{2}(X - x)\,q\left(\frac{1}{p^2} + 1\right);$$

$$\therefore\ (X - x)\,q\left(\frac{1}{p^2} + 1\right) = -\left(p + \frac{1}{p}\right) = -p\left(1 + \frac{1}{p^2}\right);$$

$$\therefore\ X - x = -\frac{p}{q};\quad X = x - \frac{p}{q} \ldots\ldots\ldots\ldots (1),$$

$$Y - y = -\frac{1-p^2}{2p}(X - x) = \frac{1-p^2}{2q};$$

$$\therefore\ Y = y + \frac{1-p^2}{2q} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

from $y = f(x)$ the equation to the curve AP, p and q may be found in terms of y and x: then between (1), (2), and $y = f(x)$, y and x may be eliminated, and the equation between Y and X or the equation to the Caustic, may be found.

Cor. 1. If the incident rays proceed from A, the origin of co-ordinates, we shall find by a similar method that the equation to the reflected ray is

$$Y - y = \frac{2px - y(1-p^2)}{2py + x(1-p^2)}(X - x).$$

Cor. 2. To find the length Pp of the reflected ray.

$$Pp^2 = (X - x)^2 + (Y - y)^2$$

$$= \frac{p^2}{q^2} + \frac{(1-p^2)^2}{4q^2} = \frac{(1+p^2)^2}{4q^2};$$

$$\therefore\ Pp = \tfrac{1}{2}\cdot\frac{1+p^2}{-q},\ \text{or, (Art. 209),}$$

the length of the reflected ray $= \frac{1}{4}$th the chord of curvature perpendicular to the axis of x.

Hence, we may construct the caustic: take OP the radius of curvature, draw $Pp = \frac{1}{2}$ the chord $\perp$ to x, and making with OP the same angle that NP does, then p will be a point in the caustic.

Ex. 1. Let the rays fall perpendicularly to the axis of the parabola.

$$y^2 = 4ax;\quad p = \frac{2a}{y};\quad q = -\frac{2a}{y^2}p = -\frac{4a^2}{y^3}.$$

$$\therefore\ X = x - \frac{p}{q} = x + \frac{y^2}{2a} = 3x;\quad \therefore\ x = \tfrac{1}{3}X,$$

$$1 - p^2 = 1 - \frac{4a^2}{y^2} = 1 - \frac{a}{x} = \frac{x-a}{x},$$

$$2q = \frac{-8a^2}{y^3} = \frac{-8a^2}{4axy} = \frac{-2a}{xy};$$

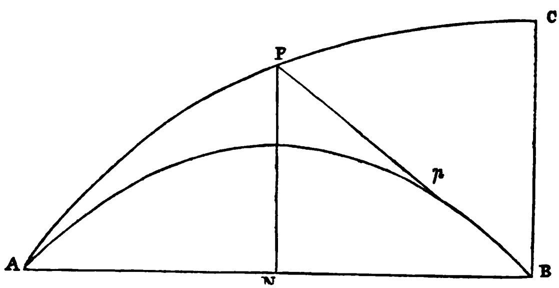
$$\therefore\ Y = y + \frac{1-p^2}{2q} = y - y\frac{x-a}{2a} = y\left(\frac{3a-x}{2a}\right);$$

$$\therefore\ Y^2 = \frac{x}{a}\cdot(3a-x)^2 = \frac{X}{27a}(9a-X)^2;$$

whence the caustic cuts the axis at the origin and at a distance $9a$ from the origin.

Ex. 2. Reflecting curve a cycloid; rays parallel to the diameter of the generating circle.

$$AN = x;\quad NP = y;\quad BC = 2a.$$



$$\text{Then } p = \frac{dy}{dx} = \frac{\sqrt{2ay - y^2}}{y},\quad q = -\frac{a}{y^2};$$

$$\therefore\ X = x - \frac{p}{q} = x + \frac{y}{a}\sqrt{2ay - y^2}\ldots\ldots\ldots\ldots(1),$$

$$Y = y + \frac{1-p^2}{2q} = \frac{1}{a}(2ay - y^2)\ldots\ldots\ldots\ldots(2);$$

$$\therefore\ y = a - \sqrt{a^2 - aY}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

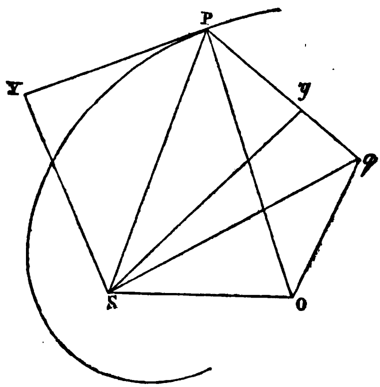
From (1) $\dfrac{dX}{dy} = \dfrac{2}{a}\sqrt{2ay - y^2} = \dfrac{2}{\sqrt{a}}\sqrt{Y}.$

From (3) $\dfrac{dy}{dY} = \dfrac{a}{2\sqrt{a^2 - aY}} = \dfrac{\sqrt{a}}{2\sqrt{a - Y}};$

$$\therefore\ \frac{dX}{dY} = \frac{\sqrt{Y}}{\sqrt{a - Y}} = \frac{Y}{\sqrt{aY - Y^2}};$$

the equation to a cycloid of which the diameter of the generating circle is a, and therefore the base $= AB$.

225. When the pole S of a spiral is the focus of incidence, to find the length Pq of the reflected ray, and the Caustic.



PO the radius of curvature $= R$,

$SP = r$, $\angle SPO = \theta = \angle OPq$,

$SY = p$, $Pq = \rho$;

q being the point of intersection of two consecutive reflected rays, and therefore a point in the caustic.

Now S, O, q may be supposed to be fixed while P moves through an indefinitely small arc.

Also $SO^2 = r^2 + R^2 - 2Rr\cos\theta$ .... (1).

$$Oq^2 = \rho^2 + R^2 - 2R\rho\cos\theta \text{ .... (2);}$$

$$\therefore\ r\frac{dr}{d\theta} - R\cos\theta\frac{dr}{d\theta} + Rr\sin\theta = 0,$$

$$\rho\frac{d\rho}{d\theta} - R\cos\theta\frac{d\rho}{d\theta} + R\rho\sin\theta = 0;$$

$$\therefore\ \frac{dr}{d\theta}(r - R\cos\theta) = -Rr\sin\theta,$$

$$\frac{d\rho}{d\theta}(\rho - R\cos\theta) = -R\rho\sin\theta.$$

But $r+\rho$, for a very small variation of P, is constant;

$\therefore\ \frac{dr}{d\theta} = -\frac{d\rho}{d\theta}$; hence by division, we have

$$\frac{\rho}{r} = \frac{R\cos\theta - \rho}{r - R\cos\theta};\quad \therefore\ \rho = \frac{Rr\cos\theta}{2r - R\cos\theta}.$$

But $R\cos\theta = \frac{1}{2}$ chord through $S = p\frac{dr}{dp}$;

$$\therefore\ \rho = \frac{pr\dfrac{dr}{dp}}{2r - p\dfrac{dr}{dp}};$$

$$\therefore\ \frac{1}{\rho} = \frac{2}{p}\cdot\frac{dp}{dr} - \frac{1}{r} = \frac{d}{dr}\left(\log\frac{p^2}{r}\right),$$

whence ρ the length of the reflected ray may be found.

226. To find the equation to the locus of q.

Join Sq; draw $Sy \perp$ to Pq.

Let $Sq = r_1$; $Sy = p_1$;

$$\therefore\ p_1 = r\sin 2\theta = 2r\sqrt{1 - \frac{p^2}{r^2}}\cdot\frac{p}{r} = \frac{2p\sqrt{r^2 - p^2}}{r} \text{ ......(1),}$$

$$\begin{aligned} r_1^2 &= r^2 + \rho^2 - 2r\rho\cos 2\theta \\ &= (r+\rho)^2 - 4r\rho\cos^2\theta \\ &= (r+\rho)^2 - \frac{4\rho p^2}{r} \text{ .... (2);} \end{aligned}$$

whence from the given equation $p = f(r)$; and from $\rho = \phi(r)$, p and ρ may be found in terms of r; and from (1) and (2) r may be eliminated, and the equation found between p_1 and r_1, which is the equation to the caustic.

Ex. 4. An indefinitely small reflector is placed in a circular ring. Every other point of the ring is luminous, find the caustic.

$$\text{Here } p = \frac{r^2}{2a}; \quad 2a = \text{diameter of ring};$$

$$\therefore \frac{p^2}{r} = \frac{r^3}{4a^2}; \quad \therefore \log\left(\frac{p^2}{r}\right) = 3\log r - \log(4a^2);$$

$$\therefore \frac{1}{\rho} = \frac{d}{dr}.\log\left(\frac{p^2}{r}\right) = \frac{3}{r}; \quad \therefore \rho = \frac{r}{3}.$$

Hence the caustic will be a circle, the diameter $\frac{2a}{3}$.

Ex. 5. Let the reflecting curve be the equiangular spiral.

$$p = mr; \quad \frac{p^2}{r} = m^2 r; \quad \log\left(\frac{p^2}{r}\right) = \log r + \log m^2;$$

$$\therefore \frac{1}{\rho} = \frac{1}{r}; \quad \rho = r,$$

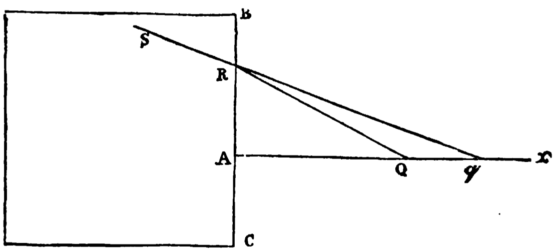
$$p_1 = 2m\sqrt{r^2 - m^2 r^2} = 2mr\sqrt{1 - m^2},$$

$$r_1^2 = 4r^2 - 4p^2 = 4r^2(1 - m^2);$$

$$\therefore p_1 = mr_1;$$

or the spiral is a similar equiangular spiral.

227. When rays of light fall upon a plane refracting surface, find the equation to the caustic.



$$\left.\begin{array}{l} QR \text{ an incident} \\ RS \text{ a refracted} \end{array}\right\} \text{ray};$$

$QA \perp$ to BAC the surface;

A the origin of co-ordinates;

$AQ = a; \quad \angle RQA = \theta; \quad \angle RqA = \phi;$

$\therefore y = -x\tan\phi + C$ is equation to Rq.

But $x = 0$; $y = AR = a\tan\theta = C$;

$$\therefore\ y = -x\tan\phi + a\tan\theta \dots\dots\dots\dots (1),$$

$$\text{also}\quad \sin\theta = m\sin\phi \dots\dots\dots\dots (2),$$

since the sines of the angles of incidence and refraction are in a constant ratio.

If now θ and ϕ be supposed to vary slightly, while y and x remain constant, the intersection p, of two of the refracted rays will be found, and p will be a point in the caustic.

$$\text{From (1)}\quad \frac{x\,d\phi}{\cos^2\phi} = \frac{a\,d\theta}{\cos^2\theta},$$

$$(2)\quad \cos\theta\,d\theta = m\cos\phi\,.\,d\phi;$$

$$\therefore\ \sqrt[3]{\frac{x}{ma}} = \frac{\cos\phi}{\cos\theta} = \frac{1}{m}\cdot\frac{\tan\theta}{\tan\phi};$$

$$\therefore\ 1 + \tan^2\phi = \frac{1}{\cos^2\phi} = 1 + \frac{1}{m^2}\tan^2\theta\sqrt[3]{\frac{m^2a^2}{x^2}}.$$

$$\text{But}\quad \frac{1}{\cos^2\phi} = (1+\tan^2\theta)\sqrt[3]{\frac{m^2a^2}{x^2}};$$

$$\therefore\ \tan^2\theta\sqrt[3]{\frac{m^2a^2}{x^2}}\,.\left\{1 - \frac{1}{m^2}\right\} = 1 - \sqrt{\frac{m^2a^2}{x^2}};$$

$$\therefore\ \tan\theta = \frac{m}{\sqrt{m^2-1}}\sqrt{\frac{x^{\frac{2}{3}}}{(ma)^{\frac{2}{3}}} - 1};$$

$$\therefore\ \tan\phi = \frac{1}{\sqrt{m^2-1}}\sqrt[3]{\frac{ma}{x}}\sqrt{\frac{x^{\frac{2}{3}}}{(ma)^{\frac{2}{3}}} - 1};$$

$$\therefore\ y = \left\{\frac{ma}{\sqrt{m^2-1}} - \frac{x^{\frac{2}{3}}\sqrt[3]{ma}}{\sqrt{m^2-1}}\right\}\sqrt{\frac{x^{\frac{2}{3}}}{(ma)^{\frac{2}{3}}} - 1}$$

$$= -\frac{ma}{\sqrt{m^2-1}}\left\{\left(\frac{x}{ma}\right)^{\frac{2}{3}} - 1\right\}^{\frac{3}{2}}.$$

$$\text{Make}\quad \frac{ma}{\sqrt{m^2-1}} = \beta, \text{ and } ma = \alpha;$$

$$\therefore\ \left(\frac{y}{\beta}\right)^{\frac{2}{3}} - \left(\frac{x}{\alpha}\right)^{\frac{2}{3}} = -1,$$

the evolute of the hyperbola, of which the centre is A and focus Q.

If m be < 1, the caustic will be the evolute of an ellipse.

CHAPTER XVI.

Change of the Independent Variable. Lagrange's Theorem.

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228. In the preceding pages, we have in general assumed, that x is the independent variable, and have derived the differential coefficients, from the equation $y=f(x)$.

We now proceed to find what values must be put for $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, &c. when y is the independent variable; and afterwards, find what must be substituted for the same quantities, when both y and x are functions of a new variable θ.

229. Prop. If $y=f(x)$; and $\therefore\ x=f^{-1}(y)$; find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, &c. in terms of $\frac{dx}{dy}$, $\frac{d^2x}{dy^2}$, &c.

Let y become $y+k$, when x becomes $x+h$;

$$\therefore\ k=\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1.2}+\frac{d^3y}{dx^3}\frac{h^3}{2.3}+\text{\&c.}\ldots\ldots\ldots(1).$$

But since $x+h=f^{-1}(y+k)$;

$$\therefore\ h=\frac{dx}{dy}k+\frac{d^2x}{dy^2}\frac{k^2}{1.2}+\frac{d^3x}{dy^3}\cdot\frac{k^3}{2.3}+\text{\&c.};$$

therefore substituting for h in equation (1), we have

$$\begin{aligned}k=&\frac{dy}{dx}\left\{\frac{dx}{dy}k+\frac{d^2x}{dy^2}\frac{k^2}{1.2}+\frac{d^3x}{dy^3}\frac{k^3}{2.3}+\text{\&c.}\right\}\\&+\frac{d^2y}{dx^2}\left\{\frac{dx^2}{dy^2}k^2+\frac{dx}{dy}\cdot\frac{d^2x}{dy^2}k^3+\text{\&c.}\right\}\cdot\frac{1}{1.2}\\&+\frac{d^3y}{dx^3}\left\{\frac{dx^3}{dy^3}k^3+\text{\&c.}\right\}\frac{1}{1.2.3}\\&+\text{\&c.}\\=&\frac{dy}{dx}\cdot\frac{dx}{dy}k+\left(\frac{dy}{dx}\cdot\frac{d^2x}{dy^2}+\frac{d^2y}{dx^2}\cdot\frac{dx^2}{dy^2}\right)\frac{k^2}{1.2}\\&+\left\{\frac{d^3x}{dy^3}\frac{dy}{dx}+3\frac{dx}{dy}\cdot\frac{d^2y}{dx^2}\cdot\frac{d^2x}{dy^2}+\frac{d^3y}{dx^3}\cdot\frac{dx^3}{dy^3}\right\}\frac{k^3}{2.3}\\&+\text{\&c.};\end{aligned}$$

$$\therefore\ 1 = \frac{dy}{dx}\cdot\frac{dx}{dy};\quad \therefore\ \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}},\ \text{(Art. 42).}$$

$$\frac{dy}{dx}\cdot\frac{d^2x}{dy^2} + \frac{d^2y}{dx^2}\cdot\frac{dx^2}{dy^2} = 0;\quad \therefore\ \frac{d^2x}{dy^2} + \frac{d^2y}{dx^3}\cdot\frac{dx^3}{dy^3} = 0;$$

$$\therefore\ \frac{d^2y}{dx^2} = \frac{-\frac{d^2x}{dy^2}}{\frac{dx^3}{dy^3}},$$

or putting

p, q, r, &c. for the differential coefficients when $y = f(x)$,

p_1, q_1, r_1, &c. .... when $x = f^{-1}(y)$;

$$p = \frac{1}{p_1};\quad q = \frac{-q_1}{p_1^3}.$$

Also, since $\dfrac{d^3x}{dy^3}\cdot\dfrac{dy}{dx} + 3\dfrac{dx}{dy}\cdot\dfrac{d^2y}{dx^2}\cdot\dfrac{d^2x}{dy^2} + \dfrac{d^3y}{dx^3}\cdot\dfrac{dx^3}{dy^3} = 0$;

$$\therefore\ r_1 p + 3p_1 q q_1 + r p_1^3 = 0,\ \text{ or }\ \frac{r_1}{p_1} - \frac{3q_1^2}{p_1^3} + r p_1^3 = 0;$$

$$\therefore\ r = \frac{3q_1^2 - r_1 p_1}{p_1^5};$$

and similarly may other coefficients be found.

230. Take the expression for the radius of curvature.

$$R = \frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}} = \frac{(1 + p^2)^{\frac{3}{2}}}{-q},\ \text{ and if } x = f^{-1}(y);$$

$$\therefore\ R = \frac{\left(1 + \frac{1}{p_1^2}\right)^{\frac{3}{2}}}{\frac{q_1}{p_1^3}} = \frac{(p_1^2 + 1)^{\frac{3}{2}}}{q_1}.$$

Ex. Let $y^2 = 4mx$; $\therefore\ x = \dfrac{y^2}{4m}$;

$$\therefore\ \frac{dx}{dy} = p_1 = \frac{y}{2m};\quad \frac{d^2x}{dy^2} = q_1 = \frac{1}{2m};$$

$$\therefore\ R = \frac{\left(1 + \frac{y^2}{4m^2}\right)^{\frac{3}{2}}}{\frac{1}{2m}} = \frac{(4m^2 + y^2)^{\frac{3}{2}}}{4m^2} = \frac{2\,.\,(m + x)^{\frac{3}{2}}}{\sqrt{m}}.$$

231. If $y=f(\theta)$, and $x=\phi(\theta)$, to express $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, &c. in terms of $\frac{dy}{d\theta}$, $\frac{dx}{d\theta}$, $\frac{d^2y}{d\theta^2}$, $\frac{d^2x}{d\theta^2}$, &c.

Let $y+k$, $\theta+m$, and $x+h$, be corresponding values of y, θ, and x; therefore, by Taylor's Theorem,

$$k=\frac{dy}{d\theta}m+\frac{d^2y}{d\theta^2}\cdot\frac{m^2}{1.2}+\frac{d^3y}{d\theta^3}\cdot\frac{m^3}{2.3}+\&c.$$

$$h=\frac{dx}{d\theta}m+\frac{d^2x}{d\theta^2}\cdot\frac{m^2}{1.2}+\frac{d^3x}{d\theta^3}\cdot\frac{m^3}{2.3}+\&c.$$

But $$k=\frac{dy}{dx}h+\frac{d^2y}{dx^2}\cdot\frac{h^2}{1.2}+\frac{d^3y}{dx^3}\cdot\frac{h^3}{2.3}+\&c.$$

$$\therefore\ \frac{dy}{d\theta}m+\frac{d^2y}{d\theta^2}\cdot\frac{m^2}{1.2}+\&c.$$

$$=\frac{dy}{dx}\cdot\left(\frac{dx}{d\theta}m+\frac{d^2x}{d\theta^2}\cdot\frac{m^2}{1.2}+\&c.\right)$$

$$+\frac{1}{1.2}\cdot\frac{d^2y}{dx^2}\left(\frac{dx^2}{d\theta^2}m^2+\frac{dx}{d\theta}\cdot\frac{d^2x}{d\theta^2}m^3+\&c.\right)+\&c.$$

$$=\frac{dy}{dx}\cdot\frac{dx}{d\theta}m+\left(\frac{dy}{dx}\cdot\frac{d^2x}{d\theta^2}+\frac{d^2y}{dx^2}\cdot\frac{dx^2}{d\theta^2}\right)\cdot\frac{m^2}{1.2}+\&c.;$$

$$\therefore\ \frac{dy}{d\theta}=\frac{dy}{dx}\cdot\frac{dx}{d\theta},\quad\text{or}\quad\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}};$$

$$\text{and}\ \frac{d^2y}{d\theta^2}=\frac{dy}{dx}\cdot\frac{d^2x}{d\theta^2}+\frac{d^2y}{dx^2}\cdot\frac{dx^2}{d\theta^2};$$

$$\therefore\ \frac{d^2y}{dx^2}=\frac{\frac{d^2y}{d\theta^2}-\frac{dy}{dx}\cdot\frac{d^2x}{d\theta^2}}{\frac{dx^2}{d\theta^2}}$$

$$=\frac{\frac{dx}{d\theta}\cdot\frac{d^2y}{d\theta^2}-\frac{dy}{d\theta}\cdot\frac{d^2x}{d\theta^2}}{\left(\frac{dx}{d\theta}\right)^3},$$

and similarly may $\frac{d^3y}{dx^3}$ be found.

232. The expression for the radius of curvature being

$$R = \frac{\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}}; \text{ if } x=\phi(\theta);\ y=f(\theta),$$

$$\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}} = \frac{\left\{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{\left(\frac{dx}{d\theta}\right)^3},$$

$$-\frac{d^2y}{dx^2} = \frac{\frac{dy}{d\theta}\cdot\frac{d^2x}{d\theta^2}-\frac{dx}{d\theta}\cdot\frac{d^2y}{d\theta^2}}{\left(\frac{dx}{d\theta}\right)^3};$$

$$\therefore\ R = \frac{\left\{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{\frac{dy}{d\theta}\cdot\frac{d^2x}{d\theta^2}-\frac{dx}{d\theta}\,\frac{d^2y}{d\theta^2}}.$$

233. Let $x=-r\cos\theta$ and $y=r\sin\theta$, find R, r being a function of θ.

$$\frac{dx}{d\theta}=r\sin\theta-\cos\theta\frac{dr}{d\theta};\quad \frac{dy}{d\theta}=r\cos\theta+\sin\theta\,.\frac{dr}{d\theta};$$

$$\frac{d^2x}{d\theta^2}=r\cos\theta+2\sin\theta\frac{dr}{d\theta}-\cos\theta\,.\frac{d^2r}{d\theta^2},$$

$$\frac{d^2y}{d\theta^2}=-r\sin\theta+2\cos\theta\frac{dr}{d\theta}+\sin\theta\,.\frac{d^2r}{d\theta^2};$$

$$\therefore\ \frac{dx^2}{d\theta^2}+\frac{dy^2}{d\theta^2}=r^2+\frac{dr^2}{d\theta^2};$$

$$\frac{dy}{d\theta}\,\frac{d^2x}{d\theta^2}-\frac{dx}{d\theta}\cdot\frac{d^2y}{d\theta^2}=r^2+2\,.\frac{dr^2}{d\theta^2}-r\,.\frac{d^2r}{d\theta^2};$$

$$\therefore\ R=\frac{\left(r^2+\frac{dr^2}{d\theta^2}\right)^{\frac{3}{2}}}{r^2+2\,.\frac{dr^2}{d\theta^2}-r\,\frac{d^2r}{d\theta^2}}.$$

Ex. 1. Let $r=a\sin\theta$, the equation to the circle from a point in the circumference, θ being the angle between the tangent at origin and r, and a the diameter;

$$\therefore\ \frac{dr}{d\theta}=a\cos\theta=\sqrt{a^2-r^2},\quad \frac{d^2r}{d\theta^2}=-a\sin\theta=-r;$$

$$\therefore\ R=\frac{(r^2+a^2-r^2)^{\frac{3}{2}}}{r^2+2a^2-2r^2+r^2}=\frac{a^3}{2a^2}=\frac{a}{2}.$$

Ex. 2. Let $r^2=a^2\cos 2\theta$, the equation to the Lemniscata;

$$\therefore\ \frac{dr}{d\theta}=-\frac{a^2}{r}.\sin 2\theta=-\frac{a^2}{r}.\sqrt{1-\frac{r^4}{a^4}};$$

$$\therefore\ r^2+\frac{dr^2}{d\theta^2}=r^2+\frac{a^4-r^4}{r^2}=\frac{a^4}{r^2},$$

$$\frac{d^2r}{d\theta^2}=\frac{a^2}{r^2}.\sin 2\theta.\frac{dr}{d\theta}-\frac{2a^2}{r}.\cos 2\theta$$

$$=-\frac{a^4}{r^3}\left(1-\frac{r^4}{a^4}\right)-\frac{2a^2}{r}.\frac{r^2}{a^2};$$

$$\therefore\ -r\frac{d^2r}{d\theta^2}=\frac{a^4-r^4}{r^2}+2r^2=\frac{a^4+r^4}{r^2};$$

$$\therefore\ r^2+2\frac{dr^2}{d\theta^2}-r\frac{d^2r}{d\theta^2}=r^2+2.\frac{a^4-r^4}{r^2}+\frac{a^4+r^4}{r^2}=\frac{3a^4}{r^2};$$

$$\therefore\ R=\frac{\left(\frac{a^4}{r^2}\right)^{\frac{3}{2}}}{\frac{3a^4}{r^2}}=\frac{a^6}{3a^4r}=\frac{a^2}{3r}.$$

Ex. 3. Transform the equation

$$\frac{d^2y}{dx^2}-\frac{x}{1-x^2}.\frac{dy}{dx}+\frac{y}{1-x^2}=0,$$

into one where $\theta=\cos^{-1}x$ is the independent variable.

$$x=\cos\theta;\ \therefore\frac{dx}{d\theta}=-\sin\theta;\ \frac{d^2x}{d\theta^2}=-\cos\theta=-x,$$

$$\frac{dy}{dx}=\frac{dy}{d\theta}.\frac{d\theta}{dx}=-\frac{1}{\sin\theta}.\frac{dy}{d\theta},$$

$$\text{and }\frac{d^2y}{dx^2}=\frac{\frac{d^2y}{d\theta^2}-\frac{dy}{dx}.\frac{d^2x}{d\theta^2}}{\frac{dx^2}{d\theta^2}}=\frac{\frac{d^2y}{d\theta^2}-\frac{1}{\sin\theta}.\cos\theta\frac{dy}{d\theta}}{\sin^2\theta};$$

$$\therefore\ \frac{1}{\sin^2\theta}.\frac{d^2y}{d\theta^2}-\frac{\cos\theta}{\sin^3\theta}.\frac{dy}{d\theta}+\frac{\cos\theta}{\sin^3\theta}.\frac{dy}{d\theta}+\frac{y}{\sin^2\theta}=0;$$

$$\therefore\ \frac{d^2y}{d\theta^2}+y=0,$$

234. Find R, the arc being the independent variable.

$$R = \frac{\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}}, \text{ and } \frac{ds^2}{dx^2} = 1 + \frac{dy^2}{dx^2}.$$

But if x and y be functions of s,

$$\frac{ds^2}{dx^2} = 1 + \frac{\left(\frac{dy}{ds}\right)^2}{\left(\frac{dx}{ds}\right)^2} = \left(\frac{ds}{dx}\right)^2 \left\{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right\};$$

$$\therefore \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1,$$

$$\text{and } \left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}} = \left(\frac{ds}{dx}\right)^3,$$

$$\text{and } -\frac{d^2y}{dx^2} = \frac{\frac{dy}{ds}\cdot\frac{d^2x}{ds^2} - \frac{dx}{ds}\cdot\frac{d^2y}{ds^2}}{\left(\frac{dx}{ds}\right)^3};$$

$$\therefore R = \frac{1}{\frac{dy}{ds}\cdot\frac{d^2x}{ds^2} - \frac{dx}{ds}\cdot\frac{d^2y}{ds^2}},$$

multiplying the numerator and denominator by ds^3,

$$R = \frac{ds^3}{dyd^2x - dxd^2y},$$

where dy, dx, d^2y, and d^2x are the first and second differentials of y and x with respect to s.

235. Again, $\because \frac{1}{R} = \frac{dy}{ds}\cdot\frac{d^2x}{ds^2} - \frac{dx}{ds}\cdot\frac{d^2y}{ds^2}$;

$$\therefore \frac{1}{R^2} = \frac{dy^2}{ds^2}\cdot\left(\frac{d^2x}{ds^2}\right)^2 - 2\frac{dy}{ds}\cdot\frac{dx}{ds}\cdot\frac{d^2x}{ds^2}\cdot\frac{d^2y}{ds^2} + \frac{dx^2}{ds^2}\cdot\left(\frac{d^2y}{ds^2}\right)^2$$

$$= \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 - \left\{\frac{dx}{ds}\cdot\frac{d^2x}{ds^2} + \frac{dy}{ds}\cdot\frac{d^2y}{ds^2}\right\}^2.$$

But $\because \frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} = 1$; $\therefore \frac{dx}{ds}\cdot\frac{d^2x}{ds^2} + \frac{dy}{ds}\cdot\frac{d^2y}{ds^2} = 0$;

$$\therefore \frac{1}{R} = \sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2};$$

$$\therefore R = \frac{1}{\sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2}}.$$

Cor. Let d^2x, and d^2y be put for the second differentials, and multiply the numerator and denominator by ds^2,

$$R = \frac{ds^2}{\sqrt{(d^2x)^2 + (d^2y)^2}}.$$

Ex. Find the radius of curvature to the catenary.

Here $x = \sqrt{c^2 + s^2}$; $y = c\log\left(\frac{s + \sqrt{s^2 + c^2}}{c}\right)$,

$$\frac{dx}{ds} = \frac{s}{\sqrt{c^2 + s^2}}; \quad \frac{d^2x}{ds^2} = \frac{c^2}{(c^2 + s^2)^{\frac{3}{2}}},$$

$$\frac{dy}{ds} = \frac{c}{\sqrt{c^2 + s^2}}; \quad \frac{d^2y}{ds^2} = \frac{-cs}{(c^2 + s^2)^{\frac{3}{2}}};$$

$$\therefore \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2x}{ds^2}\right)^2 = \frac{c^4 + c^2s^2}{(c^2 + s^2)^3} = \frac{c^2}{(c^2 + s^2)^2};$$

$$\therefore R = \frac{c^2 + s^2}{c} = \frac{x^2}{c}.$$

236. Next, let $u = f(x, y)$, to find $\frac{du}{dx}$, and $\frac{du}{dy}$ in terms of r, and θ, when $x = \phi(r, \theta)$; $y = \psi(r, \theta)$.

$$\frac{du}{dr} = \frac{du}{dx}\cdot\frac{dx}{dr} + \frac{du}{dy}\cdot\frac{dy}{dr},$$

$$\frac{du}{d\theta} = \frac{du}{dx}\cdot\frac{dx}{d\theta} + \frac{du}{dy}\cdot\frac{dy}{d\theta};$$

$$\therefore \frac{du}{dr}\cdot\frac{dy}{d\theta} - \frac{du}{d\theta}\cdot\frac{dy}{dr} = \frac{du}{dx}\cdot\left(\frac{dx}{dr}\cdot\frac{dy}{d\theta} - \frac{dy}{dr}\cdot\frac{dx}{d\theta}\right),$$

$$\frac{du}{dr}\cdot\frac{dx}{d\theta} - \frac{du}{d\theta}\cdot\frac{dx}{dr} = -\frac{du}{dy}\cdot\left(\frac{dx}{dr}\cdot\frac{dy}{d\theta} - \frac{dy}{dr}\cdot\frac{dx}{d\theta}\right);$$

$$\therefore \frac{du}{dx} = \frac{\frac{du}{dr}\cdot\frac{dy}{d\theta} - \frac{du}{d\theta}\cdot\frac{dy}{dr}}{\frac{dx}{dr}\cdot\frac{dy}{d\theta} - \frac{dy}{dr}\cdot\frac{dx}{d\theta}},$$

$$\frac{du}{dy} = -\frac{\frac{du}{dr}\cdot\frac{dx}{d\theta} - \frac{du}{d\theta}\cdot\frac{dx}{dr}}{\frac{dx}{dr}\cdot\frac{dy}{d\theta} - \frac{dy}{dr}\cdot\frac{dx}{d\theta}}.$$

237. These values are much simplified, when

$$x = r\cos\theta; \quad \text{and } y = r\sin\theta.$$

For $\frac{dx}{dr} = \cos\theta; \quad \frac{dy}{dr} = \sin\theta,$

$$\frac{dx}{d\theta} = -r\sin\theta; \quad \frac{dy}{d\theta} = r\cos\theta;$$

$$\therefore \frac{dx}{dr}\cdot\frac{dy}{d\theta} - \frac{dy}{dr}\cdot\frac{dx}{d\theta} = r(\cos^2\theta + \sin^2\theta) = r;$$

$$\therefore \frac{du}{dx} = \frac{1}{r}\left(\frac{du}{dr}\cdot r\cos\theta - \frac{du}{d\theta}\cdot\sin\theta\right)$$

$$= \frac{du}{dr}\cdot\cos\theta - \frac{du}{d\theta}\cdot\frac{\sin\theta}{r} \;\ldots\ldots\ldots\ldots (1),$$

$$\frac{du}{dy} = \frac{du}{dr}\cdot\sin\theta + \frac{du}{d\theta}\cdot\frac{\cos\theta}{r} \;\ldots\ldots\ldots\ldots (2).$$

Ex. 1. Transform $x\frac{du}{dy} - y\frac{du}{dx}$, to variables θ and r: when $x = r\cos\theta$; $y = r\sin\theta$.

$$(1) \times x = r\frac{du}{dr}\cdot\cos\theta\cdot\sin\theta + \frac{du}{d\theta}\cdot\cos^2\theta;$$

$$(2) \times y = r\frac{du}{dr}\cdot\cos\theta\cdot\sin\theta - \frac{du}{d\theta}\cdot\sin^2\theta;$$

$$\therefore x\frac{du}{dy} - y\frac{du}{dx} = \frac{du}{d\theta}\cdot(\sin^2\theta + \cos^2\theta) = \frac{du}{d\theta}.$$

Ex. 2. If $\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0$; transform it when $x = r\cos\theta$, and $y = r\sin\theta$.

From (2), $\frac{du}{dy} = \frac{du}{dr}\cdot\sin\theta + \frac{du}{d\theta}\cdot\frac{\cos\theta}{r} = V$, suppose;

$$\therefore \frac{d^2u}{dy^2} = \frac{dV}{dy} = \frac{dV}{dr}\cdot\sin\theta + \frac{dV}{d\theta}\cdot\frac{\cos\theta}{r}.$$

But $\frac{dV}{dr} = \frac{d^2u}{dr^2}\cdot\sin\theta + \frac{d^2u}{d\theta dr}\cdot\frac{\cos\theta}{r} - \frac{du}{d\theta}\cdot\frac{\cos\theta}{r^2}$,

$$\frac{dV}{d\theta} = \frac{d^2u}{drd\theta} \cdot \sin\theta + \frac{du}{dr} \cdot \cos\theta + \frac{d^2u}{d\theta^2} \cdot \frac{\cos\theta}{r} - \frac{du}{d\theta} \cdot \frac{\sin\theta}{r^2};$$

$$\therefore \frac{d^2u}{dy^2} = \sin^2\theta \cdot \frac{d^2u}{dr^2} + \frac{\cos^2\theta}{r^2} \cdot \frac{d^2u}{d\theta^2} + \frac{\cos^2\theta}{r} \cdot \frac{du}{dr}$$

$$+ \frac{2\sin\theta\cos\theta}{r} \cdot \left\{ \frac{d^2u}{d\theta dr} - \frac{1}{r} \cdot \frac{du}{d\theta} \right\}.$$

Similarly, or by changing θ into $\frac{\pi}{2} - \theta$.

$$\frac{d^2u}{dx^2} = \cos^2\theta \cdot \frac{d^2u}{dr^2} + \frac{\sin^2\theta}{r^2} \cdot \frac{d^2u}{d\theta^2} + \frac{\sin^2\theta}{r} \cdot \frac{du}{dr}$$

$$- \frac{2\sin\theta \cdot \cos\theta}{r} \cdot \left\{ \frac{d^2u}{d\theta dr} - \frac{1}{r} \cdot \frac{du}{d\theta} \right\};$$

$$\therefore \frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = \frac{d^2u}{dr^2} + \frac{1}{r^2} \cdot \frac{d^2u}{d\theta^2} + \frac{1}{r} \cdot \frac{du}{dr} = 0.$$

238. Transform the double integral $\iint V dx dy$ into one where r and θ are the variables, x and y being the same as before.

$$\because x = r\cos\theta; \quad y = r\sin\theta;$$

$$\therefore dx = \cos\theta \cdot dr - r\sin\theta \cdot d\theta,$$

$$dy = \sin\theta \cdot dr + r\cos\theta \cdot d\theta.$$

Now since in integrating, one of the quantities y or x is supposed to vary, while the other is constant, let $dx = 0$;

$$\therefore 0 = \cos\theta \cdot dr - r\sin\theta \cdot d\theta,$$

$$dy = \sin\theta \cdot dr + r\cos\theta \cdot d\theta;$$

$$\therefore \sin\theta \cdot dy = dr; \text{ eliminating } d\theta;$$

$$\therefore \text{if } dy = 0; \quad dr = 0; \quad \therefore dx = -r\sin\theta \cdot d\theta;$$

$$\therefore dxdy = -r\sin\theta \cdot d\theta \times \frac{dr}{\sin\theta} = -rdrd\theta;$$

$$\therefore \iint V dxdy = -\iint V_1 rdrd\theta.$$

Thus if $V = e^{x^2+y^2}$; $\iint e^{x^2+y^2} dxdy = -\iint e^{r^2} rdrd\theta$.

Ex. 3. If $\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0$, and $x^2 + y^2 = r^2$, transform to an equation in which r is the independent variable.

Ans. $\frac{d^2u}{dr^2} + \frac{1}{r} \cdot \frac{du}{dr} = 0.$

Ex. 4. If $\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}=0$; $r^2=x^2+y^2+z^2$,

$$\text{then } \frac{d^2u}{dr^2}+\frac{2}{r}\cdot\frac{du}{dr}=0.$$

Ex. 5. If $\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}=0$; and

$x=r\cos\theta$; $y=r\sin\theta.\sin\phi$; $z=r\sin\theta.\cos\phi$;
transform into a function of r, θ, ϕ;
assume $\rho=r\sin\theta$; and use Ex. 2;

$$\therefore\ r\frac{d^2(ru)}{dr^2}+\frac{1}{\sin^2\theta}\cdot\frac{d^2u}{d\phi^2}+\frac{1}{\sin\theta}\cdot\frac{d}{d\theta}\cdot\left(\sin\theta.\frac{du}{d\theta}\right)=0.$$

See *Camb. Math. Journal*, Vol. I. p. 121; and O'Brien's *Tracts*.

Ex. 6. Transform $\iiint Vdxdydz$, to a function of r, θ, ϕ,

$$\iiint Vdxdydz=\iiint Vr^2dr\sin\theta.d\theta.d\phi.$$

Lagrange's Theorem.

239. Let $u=f(y)$, where $y=z+x\phi(y)$, and z is independent of x; required u or $f(y)$ in terms of x.

By Maclaurin,

$$u=U_0+U_1x+U_2\frac{x^2}{1.2}+U_3\frac{x^3}{2.3}+\&c.+\frac{U_nx^n}{1.2.3\ldots n}+\&c.$$

where U_0, U_1, U_2, &c. are the values of u, $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$, &c.; when $x=0$.

First, if $x=0$, $y=z$; $\therefore\ U_0=f(z)$.

$$\text{Now } \frac{du}{dx}=\frac{du}{dy}\cdot\frac{dy}{dx},\ \text{and } \frac{du}{dz}=\frac{du}{dy}\cdot\frac{dy}{dz}.$$

But $\frac{dy}{dx}=x.\phi'(y)\cdot\frac{dy}{dx}+\phi(y)$, where $\phi'y=\frac{d\phi(y)}{dy}$;

$$\therefore\ \frac{dy}{dx}=\frac{\phi(y)}{1-x\phi'(y)},$$

$$\frac{dy}{dz}=1+x.\phi'(y)\cdot\frac{dy}{dz};\quad \therefore\ \frac{dy}{dz}=\frac{1}{1-x\phi'(y)};$$

$$\therefore \frac{dy}{dx} = \phi(y) \cdot \frac{dy}{dz};$$

$$\therefore \frac{du}{dx} = \frac{du}{dy} \cdot \phi(y) \cdot \frac{dy}{dz} = \phi(y) \cdot \frac{du}{dy} \cdot \frac{dy}{dz} = \phi(y) \cdot \frac{du}{dz}.$$

Make $x = 0$; $\therefore U_1 = \phi(z) \cdot \frac{d \cdot f(z)}{dz}$.

Next, let $\phi(y) \frac{du}{dz} = \frac{du_1}{dz}$; $\therefore \frac{du}{dx} = \frac{du_1}{dz}$;

$$\therefore \frac{d^2u}{dx^2} = \frac{d^2u_1}{dxdz} = \frac{d^2u_1}{dzdx} = \frac{d \cdot \left(\frac{du_1}{dx}\right)}{dz} = \frac{d \cdot \left\{ \phi(y) \cdot \left(\frac{du_1}{dz}\right) \right\}}{dz}$$

$$= \frac{d \cdot \left\{ \{\phi(y)\}^2 \cdot \frac{du}{dz} \right\}}{dz}$$

$$U_2 = \frac{d \left\{ \{\phi(z)\}^2 \cdot \frac{d \cdot f(z)}{dz} \right\}}{dz}.$$

And so may U_3 be found; but to find U_n,

$$\text{assume } \frac{d^{n-1}u}{dx^{n-1}} = \frac{d^{n-2} \left\{ \{\phi(y)\}^{n-1} \cdot \frac{du}{dz} \right\}}{dz^{n-2}};$$

$$\text{let } \{\phi(y)\}^{n-1} \cdot \frac{du}{dz} = \frac{du_{n-1}}{dz}; \quad \therefore \frac{d^{n-1}u}{dx^{n-1}} = \frac{d^{n-1}u_{n-1}}{dz^{n-1}};$$

$$\therefore \frac{d^n u}{dx^n} = \frac{d^n u_{n-1}}{dx \cdot dz^{n-1}} = \frac{d^n u_{n-1}}{dz^{n-1} dx} = \frac{d^{n-1} \cdot \left(\frac{du_{n-1}}{dx}\right)}{dz^{n-1}}$$

$$= \frac{d^{n-1} \cdot \left\{ \{\phi(y)\} \cdot \frac{du_{n-1}}{dz} \right\}}{dz^{n-1}} = \frac{d^{n-1} \cdot \left\{ \{\phi(y)\}^n \cdot \frac{du}{dz} \right\}}{dz^{n-1}};$$

$$\therefore U_n = \frac{d^{n-1} \cdot \left\{ \{\phi(z)\}^n \cdot \frac{d \cdot f(z)}{dz} \right\}}{dz^{n-1}}.$$

Hence if the assumption be true for $n-1$, it is true for n; and it is true for $n = 1$ and $n = 2$; therefore it is universally true, and writing Z for $\frac{d \cdot f(z)}{dz}$, we have

$$u = f(z) + \{\phi(z) \cdot Z\} \cdot \frac{x}{1} + \frac{d \cdot \{[\phi(z)]^2 \cdot Z\}}{dz} \cdot \frac{x^2}{1 \cdot 2}$$
$$+ \frac{d^2 \cdot \{[\phi(z)]^3 \cdot Z\}}{dz^2} \cdot \frac{x^3}{2 \cdot 3} + \&c. + \frac{d^{n-1} \cdot \{[\phi(z)]^n \cdot Z\}}{dz^{n-1}} \cdot \frac{x^n}{1 \cdot 2 \cdot 3 \ldots n}$$
$$+ \&c. \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

which is the theorem required.

COR. If $f(y) = y$ or y be required, then

$$f(z) = z, \text{ and } Z = \frac{df(z)}{dz} = 1;$$
$$\therefore y = z + \phi(z) \cdot \frac{x}{1} + \frac{d \cdot \{\phi(z)\}^2}{dz} \cdot \frac{x^2}{1 \cdot 2}$$
$$+ \frac{d^2\{\phi(z)\}^3}{dz^2} \cdot \frac{x^3}{2 \cdot 3} + \&c. \ldots\ldots\ldots\ldots \quad (2).$$

Ex. 1. $y^3 - ay + b = 0$; find y or the root of the cubic equation.

Here $y = \frac{b}{a} + \frac{1}{a} \cdot y^3$, and taking series (2),

$$\therefore z = \frac{b}{a}, \quad x = \frac{1}{a}, \quad \phi(y) = y^3;$$
$$\therefore \phi(z) = z^3, \quad \{\phi(z)\}^2 = z^6, \quad \{\phi(z)\}^3 = z^9, \quad \{\phi(z)\}^4 = z^{12};$$
$$\therefore \frac{d \cdot \{\phi(z)\}^2}{dz} = 6z^5, \quad \frac{d^2\{\phi(z)\}^3}{dz^2} = 8 \cdot 9z^7,$$
$$\frac{d^3\{\phi(z)\}^4}{dz^3} = 10 \cdot 11 \cdot 12z^9;$$
$$\therefore y = z + z^3 \cdot \frac{x}{1} + 6 \cdot z^5 \cdot \frac{x^2}{1 \cdot 2} + 8 \cdot 9z^7 \cdot \frac{x^3}{2 \cdot 3}$$
$$+ \frac{10 \cdot 11 \cdot 12}{2 \cdot 3 \cdot 4} \cdot z^9 \cdot x^4 + \&c.$$
$$= \frac{b}{a}\{1 + \frac{b^2}{a^3} + 3 \cdot \frac{b^4}{a^6} + 12\frac{b^6}{a^9} + 55 \cdot \frac{b^8}{a^{12}} + \&c.\}.$$

Ex. 2. In the same example, find y^n.

Here $Z = nz^{n-1}$, $\phi(z) = z^3$, and using series (1);

$\therefore \phi(z) \cdot Z = nz^{n+2}$,

$$\{\phi(z)\}^2 Z = n \cdot z^{n+5}; \quad \therefore \frac{d\{\phi(z)\}^2 \cdot Z}{dz} = n \cdot (n+5) \cdot z^{n+4},$$
$$\{\phi(z)\}^3 Z = n \cdot z^{n+8}; \quad \therefore \frac{d^2\{\phi(z)\}^3 \cdot Z}{dz^2} = n \cdot (n+8) \cdot (n+7) \cdot z^{n+6};$$

$$\therefore\ y^n = z^n + nz^{n+2}\cdot\frac{x}{1} + \frac{n(n+5)}{1\,.\,2}\cdot z^{n+4}\cdot x^2$$
$$+\frac{n\,.\,(n+8)\,.\,(n+7)}{1\,.\,2\,.\,3}\,z^{n+6}\,.\,x^3 + \&c.$$
$$=\frac{b^n}{a^n}\left\{1 + n\,.\,\frac{b^2}{a^2}\cdot\frac{1}{a} + \frac{n(n+5)}{1\,.\,2}\cdot\frac{b^4}{a^4}\cdot\frac{1}{a^2}\right.$$
$$\left.+\frac{n\,.\,(n+8)\,(n+7)}{1\,.\,2\,.\,3}\cdot\frac{b^6}{a^6}\cdot\frac{1}{a^3} + \&c.\right\}$$

Ex. 3. Find $\log y$, when $1 - y + a^y = 0$.

$$y = 1 + a^y, \text{ and } u = \log y;$$
$$\therefore\ z = 1,\ \ x = 1,\ \ \phi(y) = a^y,\ f(z) = \log z;\quad \therefore\ Z = \frac{1}{z};$$
$$\therefore\ f(z) = 0;\quad \phi(z)\,.\,Z = a^z\,.\,\frac{1}{z} = a;\quad \{\phi(z)\}^2 Z = a^{2z}\,\frac{1}{z};$$
$$\therefore\ \frac{d\{\phi(z)\}^2\,.\,Z}{dz} = 2A\,.\,a^{2z}\,.\,\frac{1}{z} - \frac{a^{2z}}{z^2} = 2Aa^2 - a^2;$$
$$\text{and } \{\phi(z)\}^3\,.\,Z = a^{3z}\,.\,\frac{1}{z}\cdot\frac{d\,.\,\{\phi(z)\}^3 Z}{dz} = 3Aa^{3z}\,.\,\frac{1}{z} - \frac{a^{3z}}{z^2},$$
$$\frac{d^2\,.\,\{\phi(z)\}^3\,.\,Z}{dz^2} = 9A^2a^{3z}\,.\,\frac{1}{z} - \frac{6A\,.\,a^{3z}}{z^2} + \frac{2a^{3z}}{z^3}$$
$$= 9A^2a^3 - 6Aa^3 + 2a^3;\quad z = 1,$$
$$= a^3(9A^2 - 6A + 2);$$
$$\therefore\ \log y = a + (2A-1)\frac{a^2}{1\,.\,2} + (9A^2 - 6A + 2)\,.\,\frac{a^3}{1\,.\,2\,.\,3} + \&c.$$

Ex. 4. Let $y = m + e\sin y$, find y.

Here $z = m$, $x = e$, $\phi(y) = \sin y$, and $f(z) = z$, $Z = 1$;

$$\therefore\ y = z + \phi(z)\,\frac{x}{1} + \frac{d\{\phi(z)\}^2}{dz}\cdot\frac{x^2}{1\,.\,2} + \frac{d^2\{\phi(z)\}^3}{dz^2}\,\frac{x^3}{2\,.\,3} + \&c.$$
$$\phi(z) = \sin z = \sin m \text{ if } x = 0;\quad \therefore\ \{\phi(z)\}^2 = \sin^2 z;$$
$$\therefore\ \frac{d\{\phi(z)\}^2}{dz} = 2\sin z\,.\,\cos z = \sin 2z = \sin 2m \text{ if } x = 0,$$
$$\{\phi(z)\}^3 = \sin^3 z;\quad \therefore\ \frac{d\{\phi(z)\}^3}{dz} = 3\sin^2 z\cos z,$$
$$\frac{d^2\{\phi(z)\}^3}{dz^2} = 6\sin z\cos^2 z - 3\sin^3 z$$
$$= 6\sin z - 9\sin^3 z = \tfrac{3}{4}(3\sin 3z - \sin z);$$

$$\therefore \; y = m + \sin m \, . \frac{e}{1} + \sin 2m \, . \frac{e^2}{1.2} + \tfrac{3}{4}(3 \sin 3m - \sin m)\frac{e^3}{2.3} + \&c.$$

Ex. 5. Let $x_1 = ay + by^2 + cy^3 + ey^4 + \&c.$, find y in term of x_1.

Here $y = \frac{x_1}{a} - \frac{b}{a} . (y^2 + \frac{c}{b} y^3 + \frac{e}{b} y^4 + \&c.)$;

$$\therefore \; z = \frac{x_1}{a}; \quad x = -\frac{b}{a}; \quad \phi(y) = y^2 + \frac{c}{b} y^3 + \frac{e}{b} y^4 + \&c.;$$

$$\therefore \; y = \frac{x_1}{a} - \frac{b}{a^3} x_1^2 + \frac{2b^2 - ac}{a^5} x_1^3 - \frac{5b^3 - 5abc + a^2 e}{a^7} x_1^4 + \&c.,$$

a general formula for the inversion of series.

Ex. 6. Let $u + \frac{du}{dx} h + \frac{d^2u}{dx^2} \frac{h^2}{1.2} + \frac{d^3u}{dx^3} \frac{h^3}{2.3} + \&c. = 0$;

find h in terms of u, and its differential coefficients.

Put p, q, r, &c. for $\frac{du}{dx}, \frac{d^2u}{dx^2}, \frac{d^3u}{dx^3}$, &c.;

$$\therefore \; h = -\frac{u}{p} - \frac{1}{p}\left(\frac{qh^2}{2} + \frac{rh^3}{2.3} + \&c.\right);$$

$$\therefore \; z = -\frac{u}{p}; \quad x = -\frac{1}{p}; \quad \phi(y) = \frac{qh^2}{2} + \frac{rh^3}{2.3} + \&c.;$$

$$\therefore \; h = -\left\{\frac{u}{p} + \frac{q}{p^3} . \frac{u^2}{2} + \frac{3q^2 - pr}{p^5} \frac{u^3}{2.3} + \&c.\right\}.$$

If a be a root of an equation $u = 0$; and x an approximate value of a, so that $x + h = a$; the preceding series may be used to find a near value of the root; and it has been thus used by Lagrange. Thus if $u = x^4 - 2x^2 + 4x - 8$,

$$h = -\left\{\frac{u}{2^2 . (x^3 - x + 1)} + \frac{3x^2 - 1}{2^4 . (x^3 - x + 1)^3} . \frac{u^2}{2} + \frac{21x^4 - 12x^2 - 6x + 3}{2^6 . (x^3 - x + 1)^5} \frac{u^3}{2.3} + \&c.\right\};$$

whence if $x = \frac{3}{2}$; $\therefore \; u = -\frac{23}{16}$; $a = x + h = 1.61$ nearly; and if 1.61 be put for x, a more correct value may be obtained.

END OF THE DIFFERENTIAL CALCULUS.

THE

INTEGRAL CALCULUS.

CHAPTER I.

1. **The** Integral Calculus is the inverse of the Differential, its object being to discover the original function from a given relation between the differential coefficients and functions of x and u. At present we shall only consider the case in which the first differential coefficient $\frac{du}{dx}$ is an *explicit* function of x, as $\phi^1(x)$, and $u = \phi(x)$ is required.

2. The process by which u is found from $\frac{du}{dx}$ is called *integration*, and when performed is expressed by prefixing the symbol $\int_x$.

Thus if $\frac{du}{dx} = \phi(x)$, $u = \int_x . \phi(x) + C$.

Also since if $\frac{du}{dx} = \phi(x)$; $\therefore\ du = \phi(x) . dx$,

u is found by prefixing the symbol $\int$, thus $u = \int \phi(x) . dx + C$, for since $\int$ is the initial letter of *summa*, the integral is said to be the sum of the differentials of the function.

Hence $\int \phi(x) . dx$, and $\int_x \phi(x)$ are identical: also since $\int du = u$, we see that $\int$ and d indicate inverse operations.

A constant quantity C, is added, since constant quantities connected with the original function by the sign $\pm$ disappear in differentiation: and therefore, when we return to the original value u, an arbitrary quantity as C is added, which must be determined by the nature of the Problem.

3. The simplest case is when $\frac{du}{dx} = ax^m$.

$$\text{Let } u = Ax^n + C;\ \therefore\ \frac{du}{dx} = nAx^{n-1} = ax^m;$$

$$\therefore\ a = nA, \text{ and } m = n - 1;\quad \therefore\ n = m + 1;$$

$$\text{and } A = \frac{a}{n} = \frac{a}{m+1};\quad \int_x ax^m = \frac{a}{m+1} . x^{m+1} + C;$$

or to integrate a monomial, *add unity to the index, divide by the index so increased, and add a constant.*

COR. 1. Also if $\frac{du}{dx} = ax^{-m} = \frac{a}{x^m}$, $u = -\frac{a}{m-1} \cdot \frac{1}{x^{m-1}} + C$, which may be derived from above by writing $-m$ for m.

COR. 2. The general formula fails when $m = -1$, for then

$$u = \frac{a \cdot x^{1-1}}{1-1} + C = \frac{a}{0} + C.$$

$$\text{But if } m = -1,\ \frac{du}{dx} = \frac{a}{x} = a \cdot \frac{1}{x}.$$

Now $\frac{a}{x} = a \cdot \frac{d \cdot (\log x)}{dx}$; $\therefore a \cdot \int_x \frac{1}{x} = a \cdot \log x + C$;

the true value of u may be thus derived from the general expression, if C be first determined.

For, suppose $u = 0$ when $x = b$;

$$\therefore 0 = \frac{ab^{m+1}}{m-1} + C, \text{ or } C = -\frac{ab^{m+1}}{m+1};$$

$$\therefore u = a \cdot \frac{x^{m+1} - b^{m+1}}{m+1} = \frac{0}{0}; \text{ if } m = -1$$

$$= a \log \frac{x}{b} = a \log x - a \log b = a \log x + C;$$

4. Since if $u = \log\{f(x)\} = \log(z)$, where $z = f(x)$,

$$\frac{du}{dx} = \frac{\frac{dz}{dx}}{z}; \quad \therefore \int_x \frac{\frac{dz}{dx}}{z} = \log(z) + C.$$

Or, if there be a fraction in which the numerator is the derivative of the denominator, the integral is the logarithm of the denominator.

Ex. 1. Let $\frac{du}{dx} = \frac{x}{1+x^2} = \frac{1}{2} \cdot \frac{2x}{1+x^2}$;

$$\therefore u = \tfrac{1}{2} \cdot \log(1+x^2) = \log\sqrt{1+x^2}.$$

Ex. 2. Let $\frac{du}{dx} = \frac{2x-1}{x^2-x+1}$; $\therefore u = \log(x^2 - x + 1)$.

5. $\therefore \frac{dp}{dx} + \frac{dq}{dx} + \frac{dr}{dx} + \&c. = \frac{d}{dx}(p + q + r + \&c.)$;

$$\therefore \int_x \left\{ \frac{dp}{dx} + \frac{dq}{dx} + \frac{dr}{dx} + \&c. \right\} = \int_x \frac{d}{dx}(p+q+r+\&c.)$$
$$= p+q+r+\&c.$$

or the integral of the sum of any number of differential coefficients = sum of the integrals of each differential coefficient.

Ex. 3. Let $\frac{du}{dx} = Ax^m + Bx^n + Cx^p + \&c.$;

$$\therefore u = A\int_x x^m + B\int_x x^n + C\int_x x^p + \&c.$$
$$= \frac{A}{m+1}x^{m+1} + \frac{B}{n+1}x^{n+1} + \frac{C}{p+1}x^{p+1} + \&c.$$

6. If $\frac{du}{dx} = z^m . \frac{dz}{dx}$, where z is a function of x, find u.

Since if $u = z^{m+1} + C$, $\frac{du}{dx} = (m+1)z^m . \frac{dz}{dx}$;

$$\therefore \int_x z^m . \frac{dz}{dx} = \frac{z^{m+1}}{m+1} + C;$$

or to integrate a function of this description, *increase the index by unity, divide by the index so increased,* and by the *differential coefficient of the quantity under the index.*

Examples of Simple Integration.

(1) Let $\frac{du}{dx} = ax^3$; $\therefore u = \frac{ax^4}{4}$.

(2) Let $\frac{du}{dx} = \frac{a}{x^2} = ax^{-2}$; $\therefore u = \frac{ax^{-1}}{-1} = -\frac{a}{x}$.

(3) Let $\frac{du}{dx} = ax^{\frac{m}{n}}$; $\therefore u = \frac{n}{m+n} . ax^{\frac{m+n}{n}}$.

(4) Let $\frac{du}{dx} = (ax^n + b)^m x^{n-1}$, $du = (ax^n + b)^m . x^{n-1}dx$.

Let $z = ax^n + b$; $\therefore dz = nax^{n-1}dx$; $\therefore du = \frac{1}{na}z^m dz$;

$$\therefore u = \frac{1}{na}\int_z z^m = \frac{z^{m+1}}{na.(m+1)} = \frac{(ax^n+b)^{m+1}}{na.(m+1)}.$$

(5) $\frac{du}{dx} = (ax+b)^m$; $\therefore u = \frac{(ax+b)^{m+1}}{a.(m+1)}$.

(6) $\dfrac{du}{dx} = (ax^n + b)^m . x^r$, m being a whole number.

Expand $(ax^n + b)^m$, multiply each term by x^r, and integrate them separately.

(7) $\dfrac{du}{dx} = \dfrac{x^m}{(a+bx)^n}$, m and n being whole numbers.

Let $a + bx = z$; $\therefore\ x = \dfrac{z-a}{b}$; $dx = \dfrac{1}{b} dz$;

$$\therefore\ \frac{x^m dx}{(a+bx)^n} = \frac{(z-a)^m}{b^m . z^n} . \frac{dz}{b} = du;$$

$$\therefore\ u = \frac{1}{b^{m+1}} . \int_z \frac{(z-a)^m}{z^n}.$$

Expand $(z-a)^m$, and integrate each term separately, first dividing by z^n.

(8) $\dfrac{du}{dx} = \dfrac{1}{x^m (a+bx)^n}$, m and n being integers.

For x put $\dfrac{1}{z}$; $\therefore\ \dfrac{dx}{dz} = -\dfrac{1}{z^2}$,

$$\therefore\ \frac{du}{dz} = \frac{du}{dx} . \frac{dx}{dz} = -\frac{1}{z^2} . \frac{du}{dx} = -\frac{z^{m+n-2}}{(az+b)^n};$$

$$\therefore\ u = -\int_z \frac{z^{m+n-2}}{(az+b)^n},$$

which resolves itself into the preceding case.

(9) $\dfrac{du}{dx} = \dfrac{1}{a+bx^2} = \dfrac{1}{b} \dfrac{1}{\frac{a}{b} + x^2} = \dfrac{1}{b}\sqrt{\dfrac{b}{a}} \dfrac{d}{dx}\left(\tan^{-1} x\sqrt{\dfrac{b}{a}}\right)$;

$$\therefore\ u = \frac{1}{\sqrt{ab}} . \tan^{-1}\left(x\sqrt{\frac{b}{a}}\right).$$

(10) $\dfrac{du}{dx} = \dfrac{1}{2+3x^2}$; $\therefore\ u = \dfrac{1}{\sqrt{6}} \tan^{-1}\left(x\sqrt{\dfrac{3}{2}}\right)$.

(11) $\dfrac{du}{dx} = \dfrac{1}{a+x}$; $\therefore\ u = \log(a+x)$.

(12) $\dfrac{du}{dx} = \dfrac{x^2}{1+x^3}$; $\therefore\ u = \log \sqrt[3]{1+x^3}$.

(13) $\dfrac{du}{dx} = (a + bx + cx^2)^m . (b + 2cx)$;

$$\therefore\ u = \frac{(a+bx+cx^2)^{m+1}}{m+1}.$$

(14) $\dfrac{du}{dx} = \dfrac{x^6}{1+x^2}$.

When the index of x in the numerator is not less than that in the denominator, divide by the denominator,

$$\frac{x^6}{1+x^2} = x^4 - x^2 + 1 - \frac{1}{1+x^2};$$

$$\therefore\ u = \frac{x^5}{5} - \frac{x^3}{3} + x - \tan^{-1} x.$$

7. Integrate the following differential coefficients.

(1) $ax^{\frac{5}{2}}$. (2) $ax^3 + bx^7 + cx^9$. (3) $(ax^3+b)^2 . x^2$.

(4) $(2ax+x^2)^7 . (a+x)$. (5) $\dfrac{x^2}{(a+bx)}$. (6) $\dfrac{3x^2+2x+1}{x^3+x^2+x+2}$.

(7) $\dfrac{1}{1+5x^2}$. (8) $\dfrac{1}{x^2(a+bx)}$. (9) $\left(\dfrac{a}{x} + \dfrac{b}{x^2} + \dfrac{c}{x^3} + \dfrac{e}{x^4}\right)$.

(10) $(ax+bx^2)^2$. (11) $\dfrac{x^4}{1+x^2}$. (12) $\dfrac{x^3}{(2+x)^2}$.

(13) $\dfrac{x^5}{1+x^2}$. (14) $\dfrac{5}{2x^4+3x^2}$.

The results may be tested by differentiation.

8. In the four succeeding chapters the functions will be integrated in the following order.

(1) Rational fractions of the form

$$\frac{Ax^{m-1} + Bx^n + Cx^p + \&c.}{A_1x^m + B_1x^{n_1} + C_1x^{p_1} + \&c.}$$

(2) Irrational quantities.

(3) Exponential and logarithmic functions.

(4) Circular functions.

The integrals will be then applied to the areas and lengths of curves; to the volumes and surfaces of solids of revolution.

CHAPTER II.

Rational Fractions.

9. Every rational fraction may be represented by

$$\frac{Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \&c.}{A_1x^n + B_1x^{n-1} + C_1x^{n-2} + \&c.},$$

for the index of x in the numerator can by division be made less by unity at least, than that of x in the denominator.

This fraction may be separated into others of a simpler form. Now the denominator may be composed 1st of simple factors all different. 2nd. Some of the factors may be equal. 3rd. It may contain quadratic factors, with impossible roots. 4th. It may be an assemblage of all these.

10. First let $\frac{U}{V}$ be a fraction where V is the product of n factors each different, so that

$$V = (x - a_1)(x - a_2)(x - a_3)\ldots(x - a_n).$$

Assume $\frac{U}{V} = \frac{A_1}{(x - a_1)} + \frac{A_2}{(x - a_2)} + \frac{A_3}{(x - a_3)} + \&c. + \frac{A_n}{(x - a_n)};$

$$\therefore\ U = A_1(x-a_2).(x-a_3)\ldots(x-a_n) + A_2.(x-a_1).(x-a_3).(x-a_4)$$
$$+ \&c. + A_n.(x - a_1)(x - a_2)\ldots(x - a_{n-1})\ldots\ldots$$

Successively make $x = a_1, a_2, a_3$, &c.; and let U_{a_1}, U_{a_2}, U_{a_3}, &c. be the corresponding values of U;

$$\therefore\ U_{a_1} = A_1(a_1 - a_2)(a_1 - a_3)\ldots(a_1 - a_n),$$

$$\text{or } A_1 = \frac{U_{a_1}}{(a_1 - a_2)(a_1 - a_3)\ldots\ldots}.$$

Similarly, $A_2 = \frac{U_{a_2}}{(a_2 - a_1)(a_2 - a_3), \&c.}$, and $A_3 = \frac{U_{a_3}}{(a_3 - a_1)(a_3 - a_2)\ldots}$,

$$\int_x\left(\frac{U}{V}\right) = A_1\int_x\frac{1}{x - a_1} + A_2\int_x\frac{1}{x - a_2} + A_3\int_x\frac{1}{x - a_3} + \&c.$$
$$= A_1\log(x - a_1) + A_2\log(x - a_2) + A_3\log(x - a_3) + \&c.$$
$$= \log(x - a_1)^{A_1}(x - a_2)^{A_2}(x - a_3)^{A_3}\ldots(x - a_n)^{A_n}.$$

11. Let some of the roots be equal, viz. m of them $=a$, or let $(x-a)^m$ be a factor of V.

Let $V=(x-a)^m Q$.

Assume $\dfrac{U}{V}=\dfrac{A}{(x-a)^m}+\dfrac{B}{(x-a)^{m-1}}+\dfrac{C}{(x-a)^{m-2}}+\&c.+\dfrac{P}{Q}$;

$\therefore\ U=AQ+\{B.(x-a)+C.(x-a)^2+\&c.\}Q+P(x-a)^m$.

Let $x=a$, and let U_a, Q_a be the values of U and Q;

$$\therefore\ U_a=AQ_a,\ \text{and}\ A=\frac{U_a}{Q_a};$$

$$\therefore\ U-\frac{U_a}{Q_a}.Q=(x-a)\{[B+C.(x-a)+D(x-a)^2+\&c.]\,Q+P(x-a)^{m-1}\}.$$

Hence, as the right-hand side of the equation is divisible by $(x-a)$, the left-hand side is also, let the division be effected, and let U^1 be the quotient;

$\therefore\ U^1=\{B+C.(x-a)+D(x-a)^2+\&c.\}Q+P.(x-a)^{m-1}$.

Again, make $x=a$, and we have $B=\dfrac{U^1_a}{Q_a}$, and proceeding in the same manner we at length arrive at P, which is either constant, or a function of x; if the latter, the case is reduced to that of the preceding article.

To illustrate these methods, we will take two examples*

Ex. 1. Integrate $\dfrac{du}{dx}=\dfrac{x^2-7x+1}{x^3-6x^2+11x-6}$

$$=\frac{x^2-7x+1}{(x-1)(x-2)(x-3)}.$$

Let $\dfrac{x^2-7x+1}{x^3-6x^2+11x-6}=\dfrac{A}{x-1}+\dfrac{B}{x-2}+\dfrac{C}{x-3}$;

$\therefore\ x^2-7x+1=A(x-2)(x-3)+B(x-1)(x-3)+C.(x-1)(x-2)$.

Let $x=1$; $\therefore\ 1-7+1=-5=A(1-2)(1-3)=2A$; $\therefore\ A=-\dfrac{5}{2}$.

$x=2$; $\therefore\ 4-14+1=-9=B(2-1)(2-3)=-B$; $\therefore\ B=9$;

$x=3$; $\therefore\ 9-21+1=-11=C(3-1)(3-2)=2C$; $\therefore\ C=-\dfrac{11}{2}$;

$$\therefore\ \int_x\frac{U}{V}=-\frac{5}{2}.\int_x\frac{1}{x-1}+9\int_x\frac{1}{x-2}-\frac{11}{2}.\int_x\frac{1}{x-3}$$

* In these and the following examples the constant will be omitted.

$$= -\frac{5}{2}\log(x-1) + 9\log(x-2) - \frac{11}{2}\log(x-3)$$

$$= \log\frac{(x-2)^9}{\sqrt{(x-1)^5(x-3)^{11}}}.$$

Ex. 2. Integrate $\dfrac{du}{dx} = \dfrac{2x-5}{(x+3)(x+1)^2}$.

Let $\dfrac{2x-5}{(x+3)(x+1)^2} = \dfrac{A}{(x+1)^2} + \dfrac{B}{x+1} + \dfrac{P}{x+3}$;

$\therefore\ 2x - 5 = A.(x+3) + B(x+1)(x+3) + P.(x+1)^2.$

Let $x = -1$; $\therefore\ -7 = A(3-1) = 2A$; $\therefore\ A = -\dfrac{7}{2}$;

$$\therefore\ 2x - 5 + \frac{7}{2}(x+3) = \frac{11x+11}{2} = \frac{11}{2}(x+1)$$

$$= B(x+1)(x+3) + P(x+1)^2;$$

$$\therefore\ \frac{11}{2} = B(x+3) + P(x+1).$$

Let $x + 1 = 0$; $\therefore\ \dfrac{11}{2} = 2B$; $\therefore\ B = \dfrac{11}{4}$;

$x + 3 = 0$; $\therefore\ \dfrac{11}{2} = -2P$; $\therefore\ P = -\dfrac{11}{4}$;

$$\therefore\ \int_x \frac{U}{V} = -\frac{7}{2}\int_x \frac{1}{(x+1)^2} + \frac{11}{4}\int_x \frac{1}{x+1} - \frac{11}{4}\int_x \frac{1}{x+3}$$

$$= \frac{7}{2}\cdot\frac{1}{x+1} + \frac{11}{4}\log(x+1) - \frac{11}{4}\log(x+3)$$

$$= \frac{7}{2}\cdot\frac{1}{x+1} + \frac{11}{4}\log\left(\frac{x+1}{x+3}\right).$$

12. Next, let V contain quadratic factors having impossible roots.

(1) Let V contain two impossible roots only, and let $(x-\alpha)^2 + \beta^2$ be the quadratic factor;

$$\therefore\ V = Q.\{(x-\alpha)^2 + \beta^2\}.$$

Assume $\therefore\ \dfrac{U}{V} = \dfrac{Mx+N}{(x-\alpha)^2+\beta^2} + \dfrac{P}{Q}$;

$$\therefore\ U = (Mx+N)Q + P\{(x-\alpha)^2 + \beta^2\}.$$

$$\text{Put } x = a + \beta\sqrt{-1}; \quad \therefore (x-a)^2 + \beta^2 = 0.$$

Then U becomes $U_1 + U_2\sqrt{-1}$, and Q becomes $Q_1 + Q_2\sqrt{-1}$.

Substituting and making the sum of the possible quantities $= 0$, and also the coefficient of $\sqrt{-1} = 0$, M and N may be found. Or if P be first found, subtract $P\{(x-a)^2 + \beta^2\}$ from each side of the equation;

$$\therefore U - P\{(x-a)^2 + \beta^2\} = (Mx + N) . Q;$$

$$\therefore Mx + N = \frac{U - P . \{(x-a)^2 + \beta^2\}}{Q} \text{ is known;}$$

$$\therefore \int_x \frac{U}{V} = \int_x \frac{Mx + N}{(x-a)^2 + \beta^2} + \int_x \frac{P}{Q}.$$

To integrate $\dfrac{du}{dx} = \dfrac{Mx + N}{(x-a)^2 + \beta^2}$, let $x - a = z$;

$$\therefore \frac{du}{dz} = \frac{du}{dx} = \frac{Mz + Ma + N}{z^2 + \beta^2} = \frac{Mz}{z^2 + \beta^2} + \frac{Ma + N}{z^2 + \beta^2};$$

$$\therefore u = M\int_z \frac{z}{z^2 + \beta^2} + (Ma + N)\int_z \frac{1}{z^2 + \beta^2}$$

$$= M \log \sqrt{z^2 + \beta^2} + \frac{Ma + N}{\beta} \tan^{-1}\left(\frac{z}{\beta}\right)$$

$$= M \log \sqrt{(x-a)^2 + \beta^2} + \frac{Ma + N}{\beta} \tan^{-1}\left(\frac{x-a}{\beta}\right).$$

COR. If $a = 0$, or $\dfrac{du}{dx} = \dfrac{Mx + N}{x^2 + \beta^2}$,

$$u = M \log \sqrt{x^2 + \beta^2} + \frac{N}{\beta} \tan^{-1} \frac{x}{\beta}.$$

Ex. 3. Let $\dfrac{du}{dx} = \dfrac{x-3}{x^3+1} = \dfrac{x-3}{(x+1)(x^2-x+1)}$;

$$\text{Let } \frac{x-3}{x^3+1} = \frac{A}{x+1} + \frac{Mx + N}{x^2 - x + 1}:$$

$$\therefore x - 3 = A(x^2 - x + 1) + (x+1)(Mx + N),$$

$$x = -1; \quad \therefore -4 = 3A, \text{ or } A = -\frac{4}{3};$$

$$\therefore x - 3 + \frac{4}{3}(x^2 - x + 1) = \frac{4x^2 - x - 5}{3} = \frac{(4x-5)(x+1)}{3}.$$

$$= (x+1)(Mx+N);$$

$$\therefore \frac{4x-5}{3} = (Mx+N);$$

$$\therefore \int_x \frac{x-3}{x^3+1} = -\frac{4}{3}\int_x \frac{1}{x+1} + \frac{1}{3}\int_x \frac{4x-5}{x^2-x+1}.$$

To integrate $\dfrac{du}{dx} = \dfrac{4x-5}{x^2-x+1} = \dfrac{4x-5}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}$.

Let $x - \frac{1}{2} = z$; $\therefore \dfrac{du}{dx} = \dfrac{du}{dz}$; and $4x - 5 = 4z - 3$;

$$\therefore u = \int_z \frac{4z-3}{z^2+\frac{3}{4}} = 4\int_z \frac{z}{z^2+\frac{3}{4}} - 3\int_z \frac{1}{z^2+\frac{3}{4}}$$

$$= 2\log\left(z^2+\frac{3}{4}\right) - 2\sqrt{3}\tan^{-1}\frac{2z}{\sqrt{3}};$$

$$\therefore \int_x \frac{x-3}{x^3+1} = -\frac{4}{3}\log(x+1) + \frac{2}{3}\log(x^2-x+1) - \frac{2}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}$$

$$= \log\left(\frac{\sqrt{x^2-x+1}}{x+1}\right)^{\frac{4}{3}} - \frac{2}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}.$$

Ex. 4. Let $\dfrac{du}{dx} = \dfrac{1}{(x+1)(x+2)^2(x^2+1)}$.

Let $\dfrac{U}{V} = \dfrac{A}{x+1} + \dfrac{B}{(x+2)^2} + \dfrac{C}{x+2} + \dfrac{P}{x^2+1}$.

$$1 = A.(x+2)^2(x^2+1) + \{B + C.(x+2)\}(x^2+1)(x+1)$$
$$+ P.(x+1)(x+2)^2,$$

$$x = -2; \quad \therefore 1 = B.5.(1-2) = -5B, \text{ i.e. } B = -\frac{1}{5},$$

$$x = -1; \quad \therefore 1 = A.2 = 2A; \quad \therefore A = \frac{1}{2}.$$

$$1 - \frac{(x+2)^2(x^2+1)}{2} + \frac{(x^2+1).(x+1)}{5} = C.(x+2)(x^2+1)(x+1)$$
$$+ P.(x+1)(x+2)^2,$$

$$\text{or } -\frac{(5x^4+18x^3+23x^2+18x+8)}{10} =$$

$$(x+2).(x+1)\{C.(x^2+1) + P(x+2)\}.$$

Divide both sides by $(x+2).(x+1)$, or x^2+3x+2;

$$\therefore -\frac{5x^2+3x+4}{10} = C(x^2+1) + P.(x+2).$$

$$\text{Let } x=-2; \quad \therefore -\frac{9}{5} = 5C; \quad \therefore C = -\frac{9}{25};$$

$$\therefore \frac{9(x^2+1)}{25} - \frac{5x^2+3x+4}{10} = -\frac{(7x^2+15x+2)}{50} = P(x+2),$$

$$\text{or } -\frac{(7x+1).(x+2)}{50} = P(x+2);$$

$$\therefore P = -\frac{7x+1}{50};$$

$$\therefore \int_x \frac{U}{V} = \frac{1}{2}.\int_x \frac{1}{x+1} - \frac{1}{5}\int_x \frac{1}{(x+2)^2} - \frac{9}{25}\int_x \frac{1}{x+2} - \frac{1}{50}.\int_x \frac{7x+1}{x^2+1}$$

$$= \frac{1}{2}\log(x+1) + \frac{1}{5}\frac{1}{x+2} - \frac{9}{25}\log(x+2)$$

$$-\frac{7}{50}\log\sqrt{x^2+1} - \frac{1}{50}\tan^{-1}x.$$

13. If there be m quadratic factors, each $=(x-a)^2+\beta^2$, assume

$$\frac{U}{V} = \frac{Mx+N}{\{(x-a)^2+\beta^2\}^m} + \frac{M_1x+N_1}{\{(x-a)^2+\beta^2\}^{m-1}} + \&c. + \frac{P}{Q};$$

$$\therefore U = \{Mx+N+(M_1x+N)[(x-a)^2+\beta^2] + \&c.\}Q$$
$$+ P\{(x-a)^2+\beta^2\}^m;$$

first find $(Mx+N)$, by putting $(x-a)^2+\beta^2=0$; subtract $(Mx+N).Q$ from U; divide both sides by $(x-a)^2+\beta^2$; then proceed similarly to find M_1 and N_1.

Ex. 5. Let $\dfrac{U}{V} = \dfrac{1}{(x^2+1)^2(x+1)}$;

$$\frac{U}{V} = \frac{Mx+N}{(x^2+1)^2} + \frac{M_1x+N_1}{x^2+1} + \frac{P}{x+1};$$

$$\therefore U = 1 = \{(Mx+N)+(M_1x+N_1)(x^2+1)\}(x+1) + P(x^2+1)^2.$$

Let $x=\sqrt{-1}$;

$$\therefore 1 = (M\sqrt{-1}+N).(\sqrt{-1}+1) = -M+M\sqrt{-1}+N\sqrt{-1}+N;$$

$\therefore N - M = 1$, and $N + M = 0$; $\therefore N = \frac{1}{2} = -M$; $\therefore M = -\frac{1}{2}$,

$$1 + \tfrac{1}{2}.(x+1)(x-1) = \frac{x^2+1}{2} = (M_1x+N_1).(x^2+1).(x+1) + P(x^2+1)^2;$$

$$\therefore \tfrac{1}{2} = (M_1x + N_1)(x+1) + P(x^2+1).$$

Let $x = \sqrt{-1}$; $\therefore +\frac{1}{2} = (M_1\sqrt{-1} + N_1)(\sqrt{-1} + 1)$

$$= -M_1 + M_1\sqrt{-1} + N_1\sqrt{-1} + N_1;$$

$\therefore N_1 - M_1 = \frac{1}{2}$, and $N_1 + M_1 = 0$; $\therefore N_1 = \frac{1}{4}$, and $M_1 = -N_1 = -\frac{1}{4}$;

$$x = -1; \quad \therefore \tfrac{1}{2} = P \times 2; \quad \therefore P = \tfrac{1}{4};$$

$$\therefore \frac{U}{V} = -\tfrac{1}{2}.\frac{x-1}{(x^2+1)^2} - \tfrac{1}{4}\frac{x-1}{x^2+1} + \tfrac{1}{4}\frac{1}{x-1}.$$

14. To integrate the fraction $\frac{x-1}{(x^2+1)^2}$, divide it into two others, $\frac{x}{(x^2+1)^2}$, and $\frac{-1}{(x^2+1)^2}$. And

$$\int_x \frac{x}{(x^2+1)^2} = \int_x (x^2+1)^{-2}.x = -\frac{(x^2+1)^{-1}}{2} = -\tfrac{1}{2}.\frac{1}{(x^2+1)};$$

but $\int_x \frac{1}{(x^2+1)^2}$ is a particular case of $\int_x \frac{1}{(x^2+1)^n}$ which has not yet been integrated.

Integration by Parts.

15. The method usually given for the integration of $\int_x \frac{1}{(x^2+1)^n}$ is called the integration by parts, which is very general in its application, and which we now proceed to explain.

$$\text{Since } \frac{d}{dx}(pq) = p\frac{dq}{dx} + q\frac{dp}{dx};$$

$$\therefore p\frac{dq}{dx} = \frac{d}{dx}(pq) - q.\frac{dp}{dx};$$

$$\therefore \int_x p\frac{dq}{dx} = pq - \int_x q.\frac{dp}{dx},$$

if any differential coefficient can be divided into two parts, one of which is a function of x as p, and the other

is the differential coefficient of a known function q; then u, the required function, is equal to the product of p and q, *minus* the integral, of q multiplied by $\frac{dp}{dx}$. The utility of this method depends upon $q\frac{dp}{dx}$ being less complicated than the original function $p\frac{dq}{dx}$.

Ex. 1. Let $\frac{du}{dx} = x^3(1+x^2)^5 = x^2x(1+x^2)^5$;

$$\therefore\ p = x^2;\quad \frac{dp}{dx} = 2x;\quad \frac{dq}{dx} = x(1+x^2)^5;\quad q = \frac{(1+x^2)^6}{12};$$

$$\begin{aligned}\therefore\ \int_x x^3(1+x^2)^5 &= \frac{x^2(1+x^2)^6}{12} - \frac{1}{6}\int_x x(1+x^2)^6\\ &= \frac{x^2(1+x^2)^6}{12} - \frac{(1+x^2)^7}{84}\\ &= \frac{(1+x^2)^6}{12}\left\{x^2 - \frac{1+x^2}{7}\right\}\\ &= \frac{(1+x^2)^6}{12}\left(\frac{6x^2-1}{7}\right).\end{aligned}$$

16. Integrate $\frac{du}{dx} = \frac{1}{(x^2+1)^n}$.

$$\frac{1}{(x^2+1)^{n-1}} = \frac{x^2+1}{(x^2+1)^n} = \frac{x^2}{(x^2+1)^n} + \frac{1}{(x^2+1)^n};$$

$$\therefore\ \int_x \frac{1}{(x^2+1)^n} = \int_x \frac{1}{(x^2+1)^{n-1}} - \int_x \frac{x^2}{(x^2+1)^n}.$$

But $\int_x \frac{x^2}{(x^2+1)^n} = \int_x x\cdot\frac{x}{(x^2+1)^n}$.

$$p = x;\quad \frac{dp}{dx} = 1;\quad \frac{dq}{dx} = \frac{x}{(x^2+1)^n};\quad q = -\frac{1}{(2n-2)(x^2+1)^{n-1}};$$

$$\therefore\ \int_x \frac{x^2}{(x^2+1)^n} = \frac{-x}{(2n-2)(x^2+1)^{n-1}} + \frac{1}{2n-2}\cdot\int_x \frac{1}{(x^2+1)^{n-1}};$$

$$\begin{aligned}\therefore\ \int_x \frac{1}{(x^2+1)^n} &= \frac{x}{(2n-2)(x^2+1)^{n-1}} + \left(1 - \frac{1}{2n-2}\right)\int_x \frac{1}{(1+x^2)^{n-1}}\\ &= \frac{x}{(2n-2)(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}\cdot\int_x \frac{1}{(1+x^2)^{n-1}}.\end{aligned}$$

Ex. 2. Let $n=4$, or let $\int_x \frac{1}{(x^2+1)^4}$ be required.

$$\int_x \frac{1}{(x^2+1)^4} = \frac{1}{6}\cdot\frac{x}{(x^2+1)^3} + \frac{5}{6}\cdot\int_x \frac{1}{(x^2+1)^3},$$

$$\int_x \frac{1}{(x^2+1)^3} = \frac{1}{4}\cdot\frac{x}{(x^2+1)^2} + \frac{3}{4}\cdot\int_x \frac{1}{(x^2+1)^2},$$

$$\int_x \frac{1}{(x^2+1)^2} = \frac{1}{2}\cdot\frac{x}{x^2+1} + \frac{1}{2}\cdot\int_x \frac{1}{x^2+1}$$

$$= \frac{1}{2}\cdot\frac{x}{x^2+1} + \frac{1}{2}\tan^{-1}x;$$

$$\therefore \int_x \frac{1}{(x^2+1)^4} = \frac{1}{6}\cdot\frac{x}{(x^2+1)^3} + \frac{5}{4.6}\cdot\frac{x}{(x^2+1)^2} + \frac{3.5}{2.4.6}\cdot\frac{x}{x^2+1}$$
$$+ \frac{3.5}{2.4.6}\tan^{-1}x.$$

17. To integrate $\frac{du}{dx} = \frac{x^m}{(1+x^2)^n}$.

$$\int_x \frac{x^m}{(1+x^2)^n} = \int_x x^{m-1}\frac{x}{(1+x^2)^n};$$

$$\therefore\ p = x^{m-1};\quad \frac{dp}{dx} = (m-1)\,.\,x^{m-2};\quad q = \frac{-1}{(2n-2)(1+x^2)^{n-1}};$$

$$\therefore \int_x \frac{x^m}{(1+x^2)^n} = \frac{-x^{m-1}}{(2n-2)(1+x^2)^{n-1}} + \frac{m-1}{2n-2}\int_x \frac{x^{m-2}}{(1+x^2)^{n-1}};$$

a formula of reduction by which the original integral may be made to depend upon

$$\int_x \frac{x}{(1+x^2)^{n_1}} \text{ or } \int \frac{1}{(1+x^2)^{n_1}},$$

according as m is an odd or even integer.

18. To integrate functions of the form

$$\frac{x^m}{(a+bx+cx^2)^n} \text{ and } \frac{1}{x^m(a+bx+cx^2)^n}.$$

In these cases the trinomial $a+bx+cx^2$ must be reduced to a binomial; and then the integration may be effected by methods already given: we will first however shew how the function may be integrated when $m=0$ and $n=1$.

19. Integrate $\frac{du}{dx} = \frac{1}{a + bx + cx^2}$.

$$\frac{1}{a + bx + cx^2} = \frac{1}{c}\frac{1}{\left(\frac{a}{c} + \frac{b}{c}x + x^2\right)}.$$

$$\text{Let } x + \frac{b}{2c} = z; \quad \therefore \frac{dx}{dz} = 1,$$

$$\text{and } x^2 + \frac{bx}{c} + \frac{a}{c} = z^2 + \frac{a}{c} - \frac{b^2}{4c^2};$$

$$\therefore \frac{du}{dz} = \frac{du}{dx} = \frac{1}{c\left(z^2 + \frac{a}{c} - \frac{b^2}{4c^2}\right)}.$$

(1) Let $\frac{a}{c} > \frac{b^2}{4c^2}$, or $4ac > b^2$;

$$\therefore u = \frac{1}{c}.\int_z \frac{1}{z^2 + \frac{4ac - b^2}{4c^2}}.$$

But $\because \int_z \frac{1}{z^2 + a^2} = \frac{1}{a}.\tan^{-1}\frac{z}{a}$;

$$\therefore = \frac{1}{c}.\frac{2c}{\sqrt{4ac - b^2}}.\tan^{-1}\left(\frac{2cz}{\sqrt{4ac - b^2}}\right)$$

$$= \frac{2}{\sqrt{4ac - b^2}}.\tan^{-1}\frac{2cx + b}{\sqrt{4ac - b^2}}.$$

(2) Let $\frac{a}{c} < \frac{b^2}{4c^2}$, make $a^2 = \frac{b^2 - 4ac}{4c^2}$;

$$\therefore u = \frac{1}{c}.\int_z \frac{1}{z^2 - a^2} = \frac{1}{2ca}.\int_z\left(\frac{1}{z - a} - \frac{1}{z + a}\right)$$

$$= \frac{1}{2ca}.\log\left(\frac{z - a}{z + a}\right)$$

$$= \frac{1}{\sqrt{b^2 - 4ac}}.\log\frac{2cx + b - \sqrt{b^2 - 4ac}}{2cx + b + \sqrt{b^2 - 4ac}}.$$

20. To integrate $\frac{x^m}{(a + bx + cx^2)^n}$.

$$\frac{x^m}{(a + bx + cx^2)^n} = \frac{1}{c^n}.\frac{x^m}{\left(x^2 + \frac{b}{c}x + \frac{a}{c}\right)^n}.$$

Let $x+\dfrac{b}{2c}=z$, or $x+a=z$, if $a=\dfrac{b}{2c}$;

$$\therefore\ x^2+\frac{b}{c}x+\frac{a}{c}=z^2+\frac{a}{c}-\frac{b^2}{4c^2}=(z^2\pm\beta^2);$$

$$\therefore\ \int_x\frac{x^m}{(a+bx+cx^2)^n}=\frac{1}{c^n}\int_z\frac{(z-a)^m}{(z^2\pm\beta^2)^n}.$$

(1) Let $\dfrac{a}{c}>\dfrac{b^2}{4c^2}$; then $\displaystyle\int_z\frac{(z-a)^m}{(z^2+\beta^2)^n}$ may be found by the method used in Art. 16.

(2) Let $\dfrac{a}{c}<\dfrac{b^2}{4c^2}$; $\therefore$ $\displaystyle\int_z\frac{(z-a)^m}{(z^2-\beta^2)^n}=\int_z\frac{(z-a)^m}{(z+\beta)^n(z-\beta)^n}$ must be integrated by partial fractions.

21. To integrate $\dfrac{1}{x^m(a+bx+cx^2)^n}$.

Let $x=\dfrac{1}{z}$; $\therefore$ $\dfrac{dx}{dz}=-\dfrac{1}{z^2}$;

$$\therefore\ \frac{du}{dz}=-\frac{1}{z^2}\cdot\frac{du}{dx}=-\frac{1}{z^2}\cdot\frac{z^{m+2n}}{(az^2+bz+c)^n};$$

$$\therefore\ u=-\int_z\frac{z^{m+2n-2}}{(az^2+bz+c)^n},$$

which is the case of the preceding article.

22. To integrate $\dfrac{du}{dx}=\dfrac{1}{x^n-1}$; and $\dfrac{du}{dx}=\dfrac{1}{x^n+1}$.

Since when n is an even number,

$$x^n-1=(x-1)(x+1)(x^2-2x\cos\frac{2\pi}{n}+1)(x^2-2x\cos\frac{4\pi}{n}+1)\ldots$$

continued to the factor $x^2-2x\cos\left(\dfrac{n-2}{n}\right)\pi+1$;

and when n is odd,

$$x^n-1=(x-1)(x^2-2x\cos\frac{2\pi}{n}+1)(x^2-2x\cos\frac{4\pi}{n}+1)\ldots$$

continued to the factor $x^2-2x\cos\dfrac{n-1}{n}\pi+1$;

and $\because$ the factors of $x^n+1=0$ are contained in

$$x^2-2x\cos\frac{2m+1}{n}\pi+1,$$

we may integrate these differential coefficients by resolving them into partial fractions, having simple and quadratic factors for their denominators.

23. Let n be even, then since

$$x^n - 1 = (x-1)(x+1)\left(x^2 - 2x\cos\frac{2m\pi}{n} + 1\right),$$

where $x^2 - 2x\cos\frac{2m\pi}{n} + 1$ represents all the quadratic factors;

$$\therefore \log(x^n - 1) = \log(x-1) + \log(x+1) + \log\left(x^2 - 2x\cos\frac{2m\pi}{n} + 1\right);$$

$$\therefore \frac{nx^{n-1}}{x^n - 1} = \frac{1}{x-1} + \frac{1}{x+1} + \frac{2x - 2\cos\frac{2m\pi}{n}}{x^2 - 2x\cos\frac{2m\pi}{n} + 1};$$

$$\therefore \frac{nx^n}{x^n - 1} = \frac{x}{x-1} + \frac{x}{x+1} + \frac{2x^2 - 2x\cos\frac{2m\pi}{n}}{x^2 - 2x\cos\frac{2m\pi}{n} + 1}.$$

Now subtract n from the left-hand side of the equation, and on the right side, *unity* from each simple factor, and *two* from each quadratic factor;

$$\therefore \frac{n}{x^n - 1} = \frac{1}{x-1} - \frac{1}{x+1} - \frac{2 - 2x\cos\frac{2m\pi}{n}}{x^2 - 2x\cos\frac{2m\pi}{n} + 1};$$

$$\therefore \int_x \frac{1}{x^n - 1} = \frac{1}{n}\log\frac{x-1}{x+1} - \frac{2}{n}\int_x \frac{1 - x\cos\frac{2m\pi}{n}}{x^2 - 2x\cos\frac{2m\pi}{n} + 1}.$$

The last integral is of the form $\int_x \frac{1 - \beta x}{x^2 - 2\beta x + 1}$, and is, if we make $x - \beta = z$, $1 - \beta^2 = \delta^2$

$$= \delta\tan^{-1}\frac{z}{\delta} - \beta\log\sqrt{z^2 + \delta^2}, \text{ or since } \delta = \sin\frac{2m\pi}{n},$$

$$\int_x \frac{1}{x^n - 1} = \frac{1}{n}\log\frac{x-1}{x+1} - \frac{2}{n}\left\{\sin\frac{2m\pi}{n}\tan^{-1}\left(\frac{x - \cos\frac{2m\pi}{n}}{\sin\frac{2m\pi}{n}}\right)\right.$$

$$-\cos\frac{2m\pi}{n}\log\sqrt{x^2-2x\cos\frac{2m\pi}{n}+1}\Big\}.$$

24. The method is the same when n is odd.

The same method applies to $\frac{du}{dx}=\frac{1}{x^n+1}$; and n odd,

$$x^n+1=(x+1)\left(x^2-2x\cos\frac{2m+1}{n}\pi+1\right);$$

$$n \text{ even, } x^n+1=\left(x^2-2x\cos\frac{2m+1}{n}\pi+1\right);$$

whence giving proper values to m, u may be found.

Examples.

(1) $\int_x \frac{1}{x-x^2}=\log\left(\frac{x}{1-x}\right)$.

(2) $\int_x \frac{1}{x-x^3}=\log\frac{x}{\sqrt{1-x^2}}$.

(3) $\int_x \frac{x}{(x+a)(x+b)}=\frac{1}{b-a}\log\left\{\frac{(x+b)^b}{(x+a)^a}\right\}$.

(4) $\int_x \frac{x^2}{(x+a)(x+b)(x+c)}=\frac{a^2}{(b-a)(c-a)}\log(x+a)$

$+\frac{b^2}{(a-b)(c-b)}\log(x+b)+\frac{c^2}{(a-c)(b-c)}\log(x+c)$.

(5) $\int_x \frac{5x-2}{x^3+6x^2+8x}=\frac{1}{4}\log\frac{(x+2)^{12}}{x(x+4)^{11}}$.

(6) $\int_x \frac{3x+1}{x^3+2x^2+x}=\log\left(\frac{x}{x+1}\right)-\frac{2}{x+1}$.

(7) $\int_x \frac{1}{1+3x+2x^2}=\log\left(\frac{2x+1}{x+1}\right)$.

(8) $\int_x \frac{x}{(x+2)(x+3)^2}=-\frac{3}{x+3}+\log\left(\frac{x+3}{x+2}\right)^2$.

(9) $\int_x \frac{x}{(x-2)(x+3)^2}=-\frac{3}{5}\frac{1}{x+3}+\frac{2}{25}\log\left(\frac{x-2}{x+3}\right)$.

(10) $\int_x \frac{x^2}{(x+2)^2(x+4)^2} = -\frac{5x+12}{x^2+6x+8} + \log\left(\frac{x+4}{x+2}\right)^2.$

(11) $\int_x \frac{x^2-5}{(x+1)^3.(x-3)} = -\frac{1}{4}.\frac{5+3x}{(x+1)^2} + \frac{1}{16}\log\left(\frac{x-3}{x+1}\right).$

(12) $\int_x \frac{x}{(x^2+3x+2)^2} = \frac{3x+4}{x^2+3x+2} + \log\left(\frac{x+1}{x+2}\right)^2.$

(13) $\int_x \frac{x^2+3x+1}{x^3+x^2-2x} = \frac{1}{3}\log\frac{(x-1)^5}{\sqrt{x^4+2x^3}}.$

(14) $\int_x \frac{2x}{(x^2+1).(x^2+3)} = \log\sqrt{\frac{x^2+1}{x^2+3}}.$

(15) $\int_x \frac{x^2}{(x^2+1).(x^2+4)} = \frac{1}{3}\left\{2\tan^{-1}\frac{x}{2} - \tan^{-1}x\right\}.$

(16) $\int_x \frac{x}{(x+1)(x+2)(x^2+1)} = \frac{2}{5}\log(x+2) - \log\sqrt{x+1}$

$$+\frac{1}{10}\{\log\sqrt{x^2+1} + 3\tan^{-1}x\}.$$

(17) $\int_x \frac{x}{(x+1)(x+2)(x^2+3)}$

$$= \frac{1}{28}\left\{\log\frac{(x+2)^8}{(x+1)^7\sqrt{x^2+3}} + 3\sqrt{3}.\tan^{-1}\frac{x}{\sqrt{3}}\right\}.$$

(18) $\int_x \frac{x^2}{x^3+5x^2+8x+4} = \frac{4}{x+2} + \log(x+1).$

(19) $\int_x \frac{2x}{x^3+x^2+x+1} = \log\frac{\sqrt{x^2+1}}{x+1} + \tan^{-1}x.$

(20) $\int_x \frac{1}{x^4+4x+3} = -\frac{1}{6}\frac{1}{x+1} + \frac{1}{9}\log\left(\frac{x+1}{\sqrt{x^2-2x+3}}\right)$

$$+\frac{1}{18\sqrt{2}}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right).$$

(21) $\int_x \frac{3x^2+x-2}{(x-1)^3.(x^2+1)} = -\frac{1}{2}\frac{1}{(x-1)^2} - \frac{5}{2}.\frac{1}{x-1}$

$$+\frac{3}{2}\log\frac{\sqrt{x^2+1}}{x-1} - \tan^{-1}x.$$

(22) $\int_x \frac{1-x+x^2}{1+x+x^2+x^3} = \frac{1}{2}\left\{\log\frac{(1+x)^3}{\sqrt{1+x^2}} - \tan^{-1}x\right\}.$

(23) $\int_x \frac{1}{x^4+x^3+x^2} = -\frac{1}{x} + \log\frac{\sqrt{x^2+x+1}}{x} + \frac{1}{\sqrt{3}} \cdot \tan^{-1}\left(\frac{x+2}{x\sqrt{3}}\right).$

(24) $\int_x \frac{1}{x(1+x)^2 \cdot (1+x+x^2)} = \frac{1}{1+x} + \log\frac{\sqrt{x^2+x^3+x^4}}{(1+x)^2}$

$$-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right).$$

(25) $\int_x \frac{1}{x^8+x^7-x^4-x^3} = \frac{2-2x-5x^2}{4(1+x)x^2}$

$$+\log\sqrt[8]{\frac{(x-1)(x+1)^9}{(1+x^2)x^8}} - \frac{1}{4}\tan^{-1}x.$$

(26) $\int_x \frac{x^4}{2+3x^2} = \frac{x^3}{9} - \frac{2x}{9} + \frac{2\sqrt{2}}{9\sqrt{3}}\tan^{-1}x\sqrt{\frac{3}{2}}.$

(27) $\int_x \frac{x}{a^4+x^4} = \frac{1}{2a^2}\tan^{-1}\left(\frac{x^2}{a^2}\right).$

(28) $\int_x \frac{x^5}{1+2x^2} = \frac{x^4}{8} - \frac{x^2}{8} + \frac{1}{8}\log\sqrt{1+2x^2}.$

(29) $\int_x \frac{x^4}{(1+x^2)^2} = \left(x^3+\frac{3x}{2}\right)\frac{1}{1+x^2} - \frac{3}{2}\tan^{-1}x.$

(30) $\int_x \frac{x^5}{(1+x^2)^3} = \left(x^2+\frac{3}{4}\right)\frac{1}{(1+x^2)^2} + \log\sqrt{1+x^2}.$

(31) $\int_x \frac{1}{x^6 \cdot (1+x^2)} = -\frac{1}{5x^5} + \frac{1}{3x^3} - \frac{1}{x} - \tan^{-1}x.$

(32) $\int_x \frac{1}{x^4(1+x^2)^2} = -\left(\frac{1}{3x^3} - \frac{5}{3x} - \frac{5x}{2}\right)\frac{1}{1+x^2} + \frac{5}{2}\tan^{-1}x.$

(33) $\int_x \frac{1}{1+x+x^2} = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right).$

(34) $\int_x \frac{1}{x^2+x-1} = \frac{1}{\sqrt{5}} \cdot \log \cdot \left(\frac{2x+1-\sqrt{5}}{2x+1+\sqrt{5}}\right).$

(35) $\int_x \frac{x^2}{a+bx+cx^2} = \frac{x}{c} - \frac{b}{c^2} \cdot \log\sqrt{a+bx+cx^2}$

$$+\left(\frac{b^2}{2c^2} - \frac{a}{c}\right) \cdot \int_x \frac{1}{a+bx+cx^2}.$$

(36) $\int_x \frac{x}{(1+x+x^2)^2} = -\frac{x+2}{3(1+x+x^2)} - \frac{2}{3\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right).$

(37) $\int_x \frac{x^5}{x^3+1} = \frac{x^3}{3} - \log \sqrt[3]{x^3+1}.$

(38) $\int_x \frac{1}{1-x^4} = \log \sqrt[4]{\frac{1+x}{1-x}} - \frac{1}{2}\tan^{-1}x.$

(39) $\int_x \frac{x^2}{x^4-a^4} = \frac{1}{4a}\log\frac{x-a}{x+a} + \frac{1}{2a}\tan^{-1}\frac{x}{a}.$

(40) $\int_x \frac{3x^2}{1-x^6} = \log\sqrt{\frac{1+x^3}{1-x^3}}.$

(41) $\int_x \frac{3x^2}{1+x^6} = \tan^{-1}(x^3).$

(42) $\int_x \frac{2x}{2x^4+2x^2+1} = \cot^{-1}\left(\frac{x^2+1}{2x}\right).$

(43) $\int_x \frac{x^2}{x^4+1} = \frac{1}{4\sqrt{2}}\log\frac{x^2-x\sqrt{2+1}}{x^2+x\sqrt{2+1}} + \frac{1}{2\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{1-x^2}.$

(44) $\int_x \frac{x^4}{x^3+1} = \frac{x^2}{2} + \frac{1}{3}\log\frac{\sqrt{x^2-x+1}}{(x+1)} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{x\sqrt{3}}{2-x}.$

(45) $\int_x \frac{1}{x^6-1} = \frac{1}{6}\log\frac{(x-1)\sqrt{x^2-x+1}}{(x+1)\sqrt{x^2+x+1}} - \frac{1}{2\sqrt{3}}\tan^{-1}\frac{x\sqrt{3}}{1-x^2}.$

(46) $\int_x \frac{x^6}{1+x^3}$ being $=0$, when $x=0$, is when $x=1$

$$=\frac{1}{3}\log 2 - \frac{3}{4} + \frac{\pi}{3\sqrt{3}}.$$

(47) $\int_x \frac{1}{x(1+x^3)} = \log\sqrt[3]{\frac{x^3}{1+x^3}}.$

(48) $\int_x \frac{1}{x(1+x^4)} = \log\sqrt[4]{\frac{x^4}{1+x^4}}.$

(49) $\int_x \frac{1}{x^4(a+bx^3)} = -\frac{1}{3ax^3} + \frac{b}{3a^2}\log\left(\frac{a+bx^3}{x^3}\right).$

(50) $\int_x \frac{1}{x(1+x^3)^2} = \frac{1}{3(1+x^3)} - \log\sqrt[3]{\frac{1+x^3}{x^3}}.$

CHAPTER III.

Irrational Quantities.

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25. THESE functions will be treated in the following order:

(1) Those which are the differential coefficients of known functions.

(2) Those which may be reduced to rational functions by means of obvious substitutions.

(3) Those which must be referred by means of *Formulas of Reduction* to known integrals.

26. I. If $u = \sin^{-1} x$,

$$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}; \quad \therefore \int_x \frac{1}{\sqrt{1-x^2}} = \sin^{-1} x,$$

$$u = \cos^{-1} x, \quad \frac{du}{dx} = -\frac{1}{\sqrt{1-x^2}}; \quad \therefore \int_x \frac{-1}{\sqrt{1-x^2}} = \cos^{-1} x,$$

$$u = V\sin^{-1} x, \quad \frac{du}{dx} = \frac{1}{\sqrt{2x-x^2}}; \quad \therefore \int_x \frac{1}{\sqrt{2x-x^2}} = V\sin^{-1} x,$$

$$u = \sec^{-1} x, \quad \frac{du}{dx} = \frac{1}{x\sqrt{x^2-1}}; \quad \therefore \int_x \frac{1}{x\sqrt{x^2-1}} = \sec^{-1} x.$$

COR. Since $\frac{d}{dx}\sin^{-1}\frac{x}{a} = \frac{1}{\sqrt{a^2-x^2}}$; $\therefore \int_x \frac{1}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$.

Similarly $\int_x \frac{1}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)$.

27. II. Next, to integrate:

$$\frac{1}{\sqrt{x^2 \pm a^2}}, \quad \frac{1}{\sqrt{x^2 \pm 2ax}}, \quad \frac{1}{x\sqrt{x^2+a^2}}, \quad \frac{1}{x\sqrt{x^2-a^2}}.$$

(1) If $\frac{du}{dx} = \frac{1}{\sqrt{x^2+a^2}}$; $\therefore du = \frac{dx}{\sqrt{x^2+a^2}}$.

Let $\sqrt{x^2+a^2} = xz$; $\therefore x^2 = \frac{a^2}{z^2-1}$,

$$2\log x = \log a^2 - \log(z^2-1);$$

$$\therefore \frac{dx}{xz} = \frac{dx}{\sqrt{x^2+a^2}} = -\frac{dz}{z^2-1} = du\,;$$

$$\therefore u = -\int_z \frac{1}{z^2-1} = \tfrac{1}{2}\cdot\int_z \left\{\frac{1}{z+1} - \frac{1}{z-1}\right\} = \tfrac{1}{2}\log\frac{z+1}{z-1}$$

$$= \tfrac{1}{2}\cdot\log\frac{\sqrt{a^2+x^2}+x}{\sqrt{a^2+x^2}-x} = \tfrac{1}{2}\cdot\log\frac{(\sqrt{a^2+x^2}+x)^2}{a^2}$$

$$= \log\frac{\sqrt{a^2+x^2}+x}{a} = \log(x+\sqrt{x^2+a^2}) + C.$$

28. Since $\dfrac{1}{\sqrt{x^2+2ax}} = \dfrac{1}{\sqrt{(x+a)^2-a^2}}$;

$$\therefore \int_x \frac{1}{\sqrt{x^2+2ax}} = \log(x+a+\sqrt{x^2+2ax}).$$

29. If $\dfrac{du}{dx} = \dfrac{1}{x\sqrt{x^2+a^2}}$; $\therefore du = \dfrac{dx}{x\sqrt{x^2+a^2}}$.

Let $\sqrt{x^2+a^2} = z$; $\therefore x^2 = z^2 - a^2$; $\therefore 2\log x = \log(z^2-a^2)$.

$$\therefore \frac{dx}{xz} = \frac{1}{x\sqrt{x^2+a^2}} = \frac{1}{z^2-a^2} = du\,;$$

$$\therefore u = \int_z \frac{1}{z^2-a^2} = \frac{1}{2a}\cdot\int_z\left(\frac{1}{z-a} - \frac{1}{z+a}\right) = \frac{1}{2a}\cdot\log\frac{z-a}{z+a}$$

$$= \frac{1}{2a}\cdot\log\frac{\sqrt{x^2+a^2}-a}{\sqrt{x^2+a^2}+a} = \frac{1}{2a}\cdot\log\frac{x^2}{(\sqrt{x^2+a^2}+a)^2}$$

$$= \frac{1}{a}\cdot\log\left(\frac{x}{\sqrt{x^2+a^2}+a}\right).$$

30. Hence, if $\dfrac{du}{dx} = \dfrac{1}{x\sqrt{a^2-x^2}}$, $u = \dfrac{1}{a}\cdot\log\left(\dfrac{x}{\sqrt{a^2-x^2}+a}\right)$.

31. $$\int_x \frac{1}{\sqrt{a+bx+cx^2}} = \frac{1}{\sqrt{c}}\int_x \frac{1}{\sqrt{\frac{a}{c}+\frac{b}{c}x+x^2}}$$

$$= \frac{1}{\sqrt{c}}\int_x \frac{1}{\sqrt{\left(x+\frac{b}{2c}\right)^2+\frac{a}{c}-\frac{b^2}{4c^2}}}$$

$$= \frac{1}{\sqrt{c}}\int_x \frac{1}{\sqrt{\left(x+\frac{b}{2c}\right)^2+\frac{4ac-b^2}{4c^2}}},$$

which being of the form $\int_x \frac{1}{\sqrt{x^2 \pm a^2}}$;

$$= \frac{1}{\sqrt{c}} \cdot \log\left\{\left(x + \frac{b}{2c} + \sqrt{\frac{a}{c} + \frac{bx}{c} + x^2}\right)\frac{2c}{\sqrt{4ac - b^2}}\right\}$$

$$= \frac{1}{\sqrt{c}} \cdot \log\left(\frac{2cx + b + 2\sqrt{c}\sqrt{a + bx + cx^2}}{\sqrt{4ac - b^2}}\right).$$

32. Also, $\int_x \frac{1}{\sqrt{a + bx - cx^2}} = \frac{1}{\sqrt{c}} \int_x \frac{1}{\sqrt{\frac{a}{c} + \frac{bx}{c} - x^2}}$

$$= \frac{1}{\sqrt{c}} \int_x \frac{1}{\sqrt{\frac{a}{c} + \frac{b^2}{4c^2} - \left(x - \frac{b}{2c}\right)^2}} = \frac{1}{\sqrt{c}} \int_x \frac{1}{\sqrt{\frac{4ac + b^2}{4c^2} - \left(x - \frac{b}{2c}\right)^2}},$$

$$= \frac{1}{\sqrt{c}} \cdot \sin^{-1} \frac{x - \frac{b}{2c}}{\sqrt{\frac{4ac + b^2}{4c^2}}} = \frac{1}{\sqrt{c}} \sin^{-1}\left(\frac{2cx - b}{\sqrt{4ac + b^2}}\right).$$

33. Integrate $\frac{du}{dx} = \frac{1}{x\sqrt{a + bx + cx^2}}$.

$$\text{Let } x = \frac{1}{z}; \quad \therefore \frac{dx}{dz} = -\frac{1}{z^2};$$

$$\therefore du = \frac{dz}{\frac{1}{z}\sqrt{a + \frac{b}{z} + \frac{c}{z^2}}} \times -\frac{1}{z^2} = \frac{-dz}{\sqrt{az^2 + bz + c}};$$

$$\text{or } u = -\int_z \frac{1}{\sqrt{az^2 + bz + c}}. \quad \text{(Art. 31).}$$

34. Integrate $\frac{du}{dx} = \frac{1}{x^2\sqrt{a + bx + cx^2}}$.

Let $x = \frac{1}{z}$; $\therefore du = \frac{z^2 dz}{\sqrt{az^2 + bz + c}} \times -\frac{1}{z^2}$;

$$\therefore u = -\int_z \frac{z}{\sqrt{az^2 + bz + c}} = -\frac{1}{a} \cdot \int_z \frac{az + \frac{b}{2} - \frac{b}{2}}{\sqrt{az^2 + bz + c}}$$

$$= -\frac{1}{a} \int_z \left\{\frac{az + \frac{b}{2}}{\sqrt{az^2 + bz + c}} - \frac{b}{2} \cdot \frac{1}{\sqrt{az^2 + bz + c}}\right\}$$

$$= -\frac{1}{a}\sqrt{az^2+bz+c} + \frac{b}{2a}\cdot\int_z \frac{1}{\sqrt{az^2+bz+c}}.$$

35. Integrate $\dfrac{du}{dx} = \dfrac{1}{(a+bx)\sqrt{c+ex}}$.

Let $z = \sqrt{c+ex}$; $\therefore\ x = \dfrac{z^2-c}{e}$; $a+bx = \dfrac{ae-bc+bz^2}{e}$;

$$\therefore\ \frac{du}{dz} = \frac{e}{(ae-bc+bz^2)z}\cdot\frac{2z}{e} = \frac{2}{b\left(z^2+\dfrac{ae-bc}{b}\right)}.$$

(1) Let $ae > bc$;

$$\therefore\ u = \frac{2}{b}\times\sqrt{\frac{b}{ae-bc}}\cdot\tan^{-1}\frac{z\sqrt{b}}{\sqrt{ae-bc}}$$

$$= \frac{2}{\sqrt{b}\sqrt{ae-bc}}\tan^{-1}\left(\frac{\sqrt{b}}{\sqrt{ae-bc}}\sqrt{c+ex}\right).$$

(2) Let $ae < bc$, and let $\dfrac{bc-ae}{b} = \beta^2$;

$$\therefore\ u = \frac{2}{b}\cdot\int_z\frac{1}{z^2-\beta^2} = \frac{1}{\beta b}\cdot\int_z\left\{\frac{1}{z-\beta}-\frac{1}{z+\beta}\right\}$$

$$= \frac{1}{\beta b}\log\frac{z-\beta}{z+\beta} = \frac{1}{\beta b}\log\frac{\sqrt{c+ex}-\beta}{\sqrt{c+ex}+\beta}.$$

36. Integrate $\dfrac{du}{dx} = \dfrac{1}{(a+bx)\sqrt{c+ex^2}}$.

Let $a+bx = \dfrac{1}{z}$; $x = \dfrac{1}{b}\left(\dfrac{1}{z}-a\right)$; $\dfrac{dx}{dz} = -\dfrac{1}{bz^2}$.

$$c+ex^2 = c+\frac{ea^2}{b^2}-\frac{2ae}{bz}+\frac{e}{b^2z^2}$$

$$= \frac{b^2c+ea^2}{b^2z^2}\cdot(\beta^2-2\alpha z+z^2)\text{ by substitution;}$$

$$\therefore\ \frac{du}{dz} = \frac{bz^2}{\sqrt{b^2c+ea^2}\sqrt{\beta^2-2\alpha z+z^2}}\cdot\frac{-1}{bz^2}$$

$$= \frac{-1}{\sqrt{b^2c+ea^2}\sqrt{\beta^2-2\alpha z+z^2}};$$

$$\therefore\ u = \frac{-1}{\sqrt{cb^2+ea^2}}\cdot\int_z\frac{1}{\sqrt{\beta^2-2\alpha z+z^2}}.$$

37. Integrate $\dfrac{du}{dx} = \dfrac{1}{(a+bx^2)\sqrt{c+ex^2}}$.

$$\text{Let } \sqrt{c+ex^2} = xz; \quad \therefore x^2 = \frac{c}{z^2-e};$$

$$\therefore dx = -\frac{zdz}{x} \cdot \frac{c}{(z^2-e)^2},$$

$$a + bx^2 = \frac{az^2 - ae + bc}{z^2-e};$$

$$\therefore du = \frac{z^2-e}{az^2-ae+bc} \cdot \frac{dx}{xz}.$$

$$= -\frac{z^2-e}{az^2-ae+bc} \cdot \frac{1}{x^2} \cdot \frac{cdz}{(z^2-e)^2}$$

$$= -\frac{dz}{az^2-ae+bc} = -\frac{1}{a} \cdot \frac{dz}{z^2 \pm \beta^2},$$

the integral will be either an angle or logarithm.

38. Integrate $\dfrac{du}{dx} = \dfrac{1}{(a+bx)\sqrt{cx^2+ex+f}}$.

$$\text{Let } z = \frac{1}{a+bx}; \quad \therefore x = \frac{1}{b}\left(\frac{1}{z} - a\right); \quad \frac{dx}{dz} = -\frac{1}{b}\,\frac{1}{z^2};$$

$$\therefore cx^2 + ex + f = \frac{a^2}{b^2z^2} \cdot (z^2 - 2\beta z + \gamma);$$

$$\therefore \frac{du}{dz} = -\frac{1}{a\sqrt{z^2 - 2\beta z + \gamma}}. \quad \text{(Art. 31).}$$

39. Integrate $\dfrac{du}{dx} = X(a+bx)^{\frac{p}{q}}$, X being a rational function of x.

$$\text{Let } a + bx = z^q; \quad \therefore x = \frac{z^q - a}{b}, \text{ and } \frac{dx}{dz} = \frac{q}{b}\,z^{q-1};$$

$$\therefore \frac{du}{dz} = Z \,.\, z^p \,.\, \frac{q}{b}\,z^{q-1} = \frac{q}{b}\,Z \,.\, z^{p+q-1};$$

where Z is the value of X, when $\dfrac{z^q - a}{b}$ is put for x.

40. Integrate $\dfrac{du}{dx} = X \,.\, (x + \sqrt{1+x^2})^{\frac{p}{q}}$, where X is either a rational function of x, or of x and $\sqrt{1+x^2}$.

$$\text{Let } x + \sqrt{1+x^2} = z^q; \quad \therefore 1 + x^2 = z^{2q} - 2xz^q + x^2;$$

$$\therefore\ x = \tfrac{1}{2}\{z^q - z^{-q}\};\quad \therefore\ \frac{dx}{dz} = \frac{q}{2}\{z^{q-1} + z^{-q-1}\} = \frac{q}{2}\,\frac{z^{2q}+1}{z^{q+1}},$$

$$\text{and } \sqrt{1+x^2} = z^q + x = \tfrac{1}{2}(z^q + z^{-q});$$

$$\therefore\ \frac{du}{dz} = \int_z Z\,.\,z^p\,.\,\frac{q}{2}\,\frac{z^{2q}+1}{z^{q+1}} = \frac{q}{2}\int_z Z\,.\,\frac{z^{p+2q}+z^p}{z^{q+1}},$$

Z being the value of X, when $\frac{1}{2}(z^q - z^{-q})$ is put for x.

41. Integrate $\dfrac{du}{dx}$ when it is either

$$\frac{1}{(1-x^m)\sqrt[2m]{2x^m-1}} \text{ or } \frac{x^{m-1}}{(1-x^m)\sqrt[2m]{2x^m-1}}.$$

In the former, make $2x^m - 1 = z^{2m}x^{2m}$;

$$\therefore\ x^{2m} - 2x^m + 1 = x^{2m}(1 - z^{2m});$$

$$\text{or } \left(\frac{1-x^m}{x^m}\right)^2 = 1 - z^{2m}\ldots\ldots\ldots(1),$$

$$\text{or } \frac{1-x^m}{x^{2m+1}}\cdot\frac{dx}{dz} = z^{2m-1}\ldots\ldots\ldots\ldots(2);$$

therefore by dividing (2) by (1),

$$\frac{1}{(1-x^m)xz}\,\frac{dx}{dz} = \frac{z^{2m-2}}{1-z^{2m-2}};$$

$$\text{i. e. } \frac{du}{dx}\cdot\frac{dx}{dz} \text{ or } \frac{du}{dz} = \frac{z^{2m-2}}{1-z^{2m}}.$$

In the latter, let $2x^m - 1 = z^{2m}$; $x^{m-1}\dfrac{dx}{dz} = z^{2m-1}$,

$$\text{and } 1 - x^m = 1 - \tfrac{1}{2}(z^{2m}+1) = \tfrac{1}{2}(1 - z^{2m});$$

$$\therefore\ \frac{x^{m-1}}{(1-x^m)z}\,\frac{dx}{dz} = \frac{du}{dz} = \frac{2z^{2m-2}}{1-z^{2m}}.$$

These formulas were rationalized by Lexell.

Binomial Differential Coefficients.

42. To integrate $\dfrac{du}{dx} = x^{m-1}(a + bx^n)^{\frac{p}{q}}$.

It may be rationalized when $\dfrac{m}{n}$ or $\dfrac{m}{n} + \dfrac{p}{q}$ is an integer.

(1) Let $a+bx^n=z^q$; $\therefore\ x^n=\dfrac{z^q-a}{b}$, $x^m=\left(\dfrac{z^q-a}{b}\right)^{\frac{m}{n}}$;

$$\therefore\ x^{m-1}\frac{dx}{dz}=\frac{q}{nb}z^{q-1}\left(\frac{z^q-a}{b}\right)^{\frac{m}{n}-1};$$

$$\therefore\ \frac{du}{dz}=\frac{du}{dx}\cdot\frac{dx}{dz}=z^p\,.\,x^{m-1}\frac{dx}{dz}=\frac{q}{nb}z^{p+q-1}\left(\frac{z^q-a}{b}\right)^{\frac{m}{n}-1},$$

which is rational if $\dfrac{m}{n}$ be an integer, and easily integrable, by expanding the binomial.

(2) If $\dfrac{m}{n}$ be a fraction. Let $a+bx^n=x^nz^q$;

$$\therefore\ x^n=\frac{a}{z^q-b},\quad x^m=\frac{a^{\frac{m}{n}}}{(z^q-b)^{\frac{m}{n}}};$$

$$\therefore\ x^{m-1}\frac{dx}{dz}=-\frac{qa^{\frac{m}{n}}}{n}\cdot\frac{z^{q-1}}{(z^q-b)^{\frac{m}{n}+1}};\quad (a+bx^n)^{\frac{p}{q}}=\frac{a^{\frac{p}{q}}z^p}{(z^q-b)^{\frac{p}{q}}};$$

$$\therefore\ \frac{du}{dz}=\frac{a^{\frac{p}{q}}z^p}{(z^q-b)^{\frac{p}{q}}}\cdot x^{m-1}\cdot\frac{dx}{dz}=-\frac{qa^{\frac{m}{n}+\frac{p}{q}}}{n}\cdot\frac{z^{p+q-1}}{(z^q-b)^{\frac{m}{n}+\frac{p}{q}+1}},$$

which is rational when $\dfrac{m}{n}+\dfrac{p}{q}$ is an integer, and easily integrable if $\dfrac{m}{n}+\dfrac{p}{q}$ be a *negative* integer.

43. We have assumed that m and n are integers, but if they be fractions as $\dfrac{r}{s}$ and $\dfrac{s_1}{r_1}$. Make $v^{ss_1}=x$; $\therefore\ x^{\frac{r}{s}}=v^{rs_1}$, and $x^{\frac{r_1}{s_1}}=v^{r_1s}$. Also n is assumed positive, for if not, let

$$x=\frac{1}{u};\ \therefore\ x^{-n}=u^n.$$

Ex. 1. Let $\dfrac{du}{dx}=x^3\sqrt{1+x^2}$.

Here $m-1=3$, and $n=2$; $\therefore\ \dfrac{m}{n}=\dfrac{4}{2}=2$.

Let $1+x^2=z^2$; $\therefore\ x^2=z^2-1$, $x^4=(z^2-1)^2$;

$$\therefore\ x^3\frac{dx}{dz}=(z^2-1)z;$$

$$\therefore \frac{du}{dz} = x^3 z \frac{dx}{dz} = z^2(z^2-1) = z^4 - z^2;$$

$$\therefore u = \frac{z^5}{5} - \frac{z^3}{3} + C = \frac{(1+x^2)^{\frac{5}{2}}}{5} - \frac{(x^2+1)^{\frac{3}{2}}}{3} + C.$$

Ex. 2. Let $\dfrac{du}{dx} = \dfrac{1}{x^4\sqrt{1+x^2}}.$

Here $\dfrac{m}{n} = -\dfrac{3}{2}$, and $\dfrac{p}{q} = -\frac{1}{2}$; $\therefore \dfrac{m}{n} + \dfrac{p}{q} = -2.$

$$\text{And } \frac{1}{x^4\sqrt{1+x^2}} = \frac{1}{x^5\sqrt{x^{-2}+1}} = \frac{x^{-5}}{\sqrt{x^{-2}+1}}.$$

Let $x^{-2}+1 = z^2$; $\therefore x^{-3} = -z \cdot \dfrac{dz}{dx}$, $x^{-2} = z^2 - 1$;

$$\therefore \frac{du}{dz} = \frac{z^2-1}{z} x^{-3} \frac{dx}{dz} = -(z^2-1);$$

$$\therefore u = -\left(\frac{z^3}{3} - z\right) = -z \cdot \left(\frac{z^2}{3} - 1\right)$$

$$= -\frac{\sqrt{1+x^2}}{x}\left\{\frac{1+x^2}{3x^2} - 1\right\} = \frac{(2x^2-1)\sqrt{1+x^2}}{3x^3}.$$

Ex. 3. Let $\dfrac{du}{dx} = \dfrac{1}{\sqrt[n]{1-x^n}} = (1-x^n)^{-\frac{1}{n}}.$

$$\text{Here } \frac{m}{n} + \frac{p}{q} = \frac{1}{n} - \frac{1}{n} = 0.$$

Let $\therefore 1 - x^n = x^n z^n$; $\therefore x^n = \dfrac{1}{1+z^n}$;

$$\therefore n \log x = -\log(1+z^n); \quad \therefore \frac{1}{x}\frac{dx}{dz} = -\frac{z^{n-1}}{1+z^n};$$

$$\therefore \frac{du}{dz} = \frac{1}{xz} \cdot \frac{dx}{dz} = -\frac{z^{n-2}}{1+z^n},$$

which may be integrated by partial fractions.

44. This method of substitution is seldom adopted, the integration by parts being more generally useful, we shall henceforth confine our attention chiefly to it.

45. Ex. $\dfrac{du}{dx} = \dfrac{x^m}{\sqrt{1-x^2}}$;

$$\because \frac{x^m}{\sqrt{1-x^2}} = x^{m-1}\frac{x}{\sqrt{1-x^2}}; \quad \therefore p = x^{m-1}, \quad \frac{dq}{dx} = \frac{x}{\sqrt{1-x^2}};$$

$$\therefore \frac{dp}{dx} = (m-1)x^{m-2}; \quad q = -\sqrt{1-x^2};$$

$$\therefore\ u = -x^{m-1}\sqrt{1-x^2} + (m-1)\int_x x^{m-2}\sqrt{1-x^2}$$

$$= -x^{m-1}\sqrt{1-x^2} + (m-1)\int_x \frac{x^{m-2}}{\sqrt{1-x^2}} - (m-1)u^*;$$

$$\therefore\ m\int_x \frac{x^m}{\sqrt{1-x^2}} = -x^{m-1}\sqrt{1-x^2} + (m-1)\int_x \frac{x^{m-2}}{\sqrt{1-x^2}};$$

$$\therefore\ \int_x \frac{x^m}{\sqrt{1-x^2}} = -\frac{x^{m-1}\sqrt{1-x^2}}{m} + \frac{m-1}{m}\int_x \frac{x^{m-2}}{\sqrt{1-x^2}};$$

by putting $m-2$, $m-4$, &c. for m, the integral may be reduced either to

$$\int_x \frac{x}{\sqrt{1-x^2}},\ \text{ or } \int_x \frac{1}{\sqrt{1-x^2}};$$

or, to $-\sqrt{1-x^2}$, or $\sin^{-1}x$, according as m is odd or even.

Ex. Let $\int_x \frac{x^4}{\sqrt{1-x^2}}$ be required. Here $m = 4$.

$$\int_x \frac{x^4}{\sqrt{1-x^2}} = -\frac{x^3\sqrt{1-x^2}}{4} + \tfrac{3}{4}\int_x \frac{x^2}{\sqrt{1-x^2}};$$

$$\int_x \frac{x^2}{\sqrt{1-x^2}} = -\frac{x\sqrt{1-x^2}}{2} + \tfrac{1}{2}\int_x \frac{1}{\sqrt{1-x^2}}$$

$$= -\frac{x\sqrt{1-x^2}}{2} + \tfrac{1}{2}\sin^{-1}x + C;$$

$$\therefore\ \int_x \frac{x^4}{\sqrt{1-x^2}} = -\sqrt{1-x^2}\left\{\frac{x^3}{4} + \frac{3x}{2.4}\right\} + \frac{1.3}{2.4}\sin^{-1}x + C.$$

46. To find the general value of the integral.

(1) Let m be even $= 2n$,

let $P_{2n} = \int_x \frac{x^{2n}}{\sqrt{1-x^2}}$, and $Q_{2n-1} = x^{2n-1}\sqrt{1-x^2}$;

$$\therefore\ P_{2n} = -\frac{1}{2n}Q_{2n-1} + \frac{2n-1}{2n}P_{2n-2},$$

$$P_{2n-2} = -\frac{1}{2n-2}Q_{2n-3} + \frac{2n-3}{2n-2}P_{2n-4},$$

$$P_{2n-4} = -\frac{1}{2n-4}Q_{2n-5} + \frac{2n-5}{2n-4}P_{2n-6},$$

$$\vdots$$

$$P_2 = -\tfrac{1}{2}Q_1 + \tfrac{1}{2}P_0,\ \text{ where } P_0 = \sin^{-1}x;$$

* Since $x^{m-2}\sqrt{1-x^2} = \frac{x^{m-2}}{\sqrt{1-x^2}} - \frac{x^m}{\sqrt{1-x^2}}$.

$$\therefore\ P_{2n} = -\left\{ \frac{Q_{2n-1}}{2n} + \frac{2n-1}{2n.(2n-2)} Q_{2n-3} \right.$$

$$\left. + \frac{(2n-1).(2n-3)}{2n.(2n-2)(2n-4)} Q_{2n-5} + \&c. \right\}$$

$$+ \frac{(2n-1)(2n-3)(2n-5)......3.1}{2n.(2n-2)(2n-4)......4.2} \sin^{-1} x + C.$$

If the integral $= 0$, when $x = 0$. Then $C = 0$, for Q_{2n-1}, Q_{2n-3}, &c. each $= 0$.

If $x = 1$, Q_{2n-1}, Q_{2n-3}, &c. each $= 0$, and $\sin^{-1} x = \frac{\pi}{2}$;

$$\therefore \int_x \frac{x^{2n}}{\sqrt{1-x^2}}, \left.\begin{matrix}\text{from } x=0 \\ \text{to } x=1\end{matrix}\right\} = \frac{(2n-1).(2n-3).(2n-5)...3.1}{2n.(2n-2).(2n-4)...4.2} \cdot \frac{\pi}{2}.$$

(2) Let m be odd and $= 2n+1$;

$$\therefore\ P_{2n+1} = -\frac{1}{2n+1} Q_{2n} + \frac{2n}{2n+1} P_{2n-1},$$

$$P_{2n-1} = -\frac{1}{2n-1} Q_{2n-2} + \frac{2n-2}{2n-1} P_{2n-3},$$

$$\vdots$$

$$P_3 = -\tfrac{1}{3} Q_2 + \tfrac{2}{3} P_1,$$

$$\text{and } P_1 = \int_x \frac{x}{\sqrt{1-x^2}} = -\sqrt{1-x^2};$$

$$\therefore\ P_{2n+1} = -\left\{ \frac{1}{2n+1} Q_{2n} + \frac{2n}{(2n+1)(2n-1)} Q_{2n-2} \right.$$

$$\left. + \frac{2n.(2n-2)}{(2n+1)(2n-1)(2n-3)} Q_{2n-4} + \&c. \right\}$$

$$- \frac{2n.(2n-2)(2n-4)...4.2}{(2n+1)(2n-1)(2n-3)...5.3} \sqrt{1-x^2} + C.$$

If $P_{2n+1} = 0$ when $x = 0$, since then $Q_{2n} = 0$;

$$\therefore\ 0 = -\frac{2n.(2n-2)(2n-4)...4.2}{(2n+1)(2n-1)(2n-3)...5.3} + C;$$

whence by subtraction,

$$P_{2n+1} = \frac{2n.(2n-2)(2n-4)...4.2}{(2n+1)(2n-1)(2n-3)...5.3}$$

$$-\left\{ \frac{1}{2n+1} Q_{2n} + \frac{2n}{(2n+1)(2n-1)} Q_{2n-2} + \&c. \right\}.$$

Let $x = 1$;

$$\therefore \int_x \frac{x^{2n+1}}{\sqrt{1-x^2}}, \begin{matrix}\text{from } x=0 \\ \text{to } x=1\end{matrix}\Bigg\} = \frac{2n.(2n-2)...6.4.2}{(2n+1)(2n-1)...7.5.3}.$$

COR. If n be infinite, we may make $P_{2n} = P_{2n+1}$,

$$\text{or } \frac{\pi}{2}.\frac{1.3.5.7, \text{ \&c.}}{2.4.5.8, \text{ \&c.}} = \frac{2.4.6.8, \text{ \&c.}}{3.5.7.9, \text{ \&c.}},$$

$$\text{or } \frac{\pi}{2} = \frac{2.2.4.4.6.6.8.8, \text{ \&c.}}{1.3.3.5.5.7.7.9, \text{ \&c.}},$$

which is Wallis's Theorem for the length of the circle.

47. Let $\dfrac{du}{dx} = (a^2 - x^2)^{\frac{n}{2}}$.

$$(a^2 - x^2)^{\frac{n}{2}} = a^2.(a^2-x^2)^{\frac{n-2}{2}} - x^2.(a^2-x^2)^{\frac{n-2}{2}};$$

$$\therefore u = a^2 \int_x (a^2-x^2)^{\frac{n-2}{2}} - \int_x x^2.(a^2-x^2)^{\frac{n-2}{2}}$$

$$= a^2 \int_x (a^2-x^2)^{\frac{n-2}{2}} + \frac{x(a^2-x^2)^{\frac{n}{2}}}{n} - \frac{u}{n};$$

$$\therefore u = \frac{x(a^2-x^2)^{\frac{n}{2}}}{n+1} + \frac{na^2}{n+1}.\int_x (a^2-x^2)^{\frac{n-2}{2}};$$

by which u is reduced, n being odd to $\int_x (a^2-x^2)^{\frac{1}{2}}$.

$$\text{Also } \int_x (a^2-x^2)^{\frac{1}{2}} = \int_x \frac{a^2}{\sqrt{a^2-x^2}} - \int_x \frac{x^2}{\sqrt{a^2-x^2}}$$

$$= \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}.\sin^{-1}.\frac{x}{a}.$$

If the integral be required between $x = 0$ and $x = a$;

$$\int_x (a^2-x^2)^{\frac{1}{2}} = \frac{\pi a^2}{4} = \frac{1}{2}.\frac{\pi a^2}{2};$$

$$\therefore \int_x (a^2-x^2)^{\frac{3}{2}} = \frac{3a^2}{4}\int (a^2-x^2)^{\frac{1}{2}} = \frac{1.3}{2.4}.\frac{\pi a^4}{2}$$

$$\int_x (a^2-x^2)^{\frac{5}{2}} = \frac{5a^2}{6}.\int (a^2-x^2)^{\frac{3}{2}} = \frac{1.3.5}{2.4.6}.\frac{\pi a^6}{2};$$

$$\text{and } \int_x (a^2-x^2)^{\frac{n}{2}} = \frac{1.3.5.7...(n-2).n}{2.4.6.8...(n-1)(n+1)}.\frac{\pi a^{n+1}}{2}.$$

48. Integrate $\dfrac{du}{dx}=\dfrac{1}{x^m\sqrt{1+x^2}}$.

$$\int_x \frac{1}{x^m\sqrt{1+x^2}}=\int_x \frac{x}{x^{m+1}\sqrt{1+x^2}}.$$

$$\text{Here } p=\frac{1}{x^{m+1}},\text{ and } \frac{dq}{dx}=\frac{x}{\sqrt{1+x^2}};$$

$$\therefore \frac{dp}{dx}=-\frac{m+1}{x^{m+2}},\text{ and } q=\sqrt{1+x^2};$$

$$\therefore \int_x \frac{1}{x^m\sqrt{1+x^2}}=\frac{\sqrt{1+x^2}}{x^{m+1}}+(m+1)\,.\int_x \frac{\sqrt{1+x^2}}{x^{m+2}}$$

$$=\frac{\sqrt{1+x^2}}{x^{m+1}}+(m+1)\,.\int_x \frac{1}{x^{m+2}\sqrt{1+x^2}}+(m+1)\,.\int_x \frac{1}{x^m\sqrt{1+x^2}};$$

$$\therefore \int_x \frac{1}{x^{m+2}\sqrt{1+x^2}}=-\frac{1}{m+1}\cdot\frac{\sqrt{1+x^2}}{x^{m+1}}-\frac{m}{m+1}\cdot\int_x \frac{1}{x^m\sqrt{1+x^2}}.$$

For $m+2$ put m;

$$\therefore \int_x \frac{1}{x^m\sqrt{1+x^2}}=-\frac{1}{m-1}\cdot\frac{\sqrt{1+x^2}}{x^{m-1}}-\frac{m-2}{m-1}\cdot\int_x \frac{1}{x^{m-2}\sqrt{1+x^2}};$$

and the integral may be reduced either to $\displaystyle\int_x \frac{1}{x\sqrt{1+x^2}}$, or $\displaystyle\int_x \frac{1}{x^2\sqrt{1+x^2}}$, according as m is odd or even;

also $\displaystyle\int_x \frac{1}{x\sqrt{1+x^2}}=\log\frac{x}{1+\sqrt{1+x^2}}$,

and $\displaystyle\int_x \frac{1}{x^2\sqrt{1+x^2}}=\int_x \frac{1}{x^3\sqrt{x^{-2}+1}}=-\sqrt{x^{-2}+1}=-\frac{\sqrt{1+x^2}}{x}$.

49. Integrate $\dfrac{du}{dx}=\dfrac{1}{x^m\sqrt{x^2-1}}$.

$$\int_x \frac{1}{x^m\sqrt{x^2-1}}=\int_x \frac{x}{x^{m+1}\sqrt{x^2-1}}=\int_x \frac{1}{x^{m+1}}\cdot\frac{d\,.\sqrt{x^2-1}}{dx}$$

$$=\frac{\sqrt{x^2-1}}{x^{m+1}}+(m+1)\,.\int_x \frac{\sqrt{x^2-1}}{x^{m+2}}$$

$$=\frac{\sqrt{x^2-1}}{x^{m+1}}+(m+1)\,.\int_x \frac{1}{x^m\sqrt{x^2-1}}-(m+1)\,.\int_x \frac{1}{x^{m+2}\sqrt{x^2-1}};$$

$$\therefore \int_x \frac{1}{x^{m+2}\sqrt{x^2-1}} = \frac{1}{m+1}\cdot\frac{\sqrt{x^2-1}}{x^{m+1}} + \frac{m}{m+1}\cdot\int_x \frac{1}{x^m\sqrt{x^2-1}};$$

therefore, writing m for $(m+2)$,

$$\int_x \frac{1}{x^m\sqrt{x^2-1}} = \frac{1}{m-1}\cdot\frac{\sqrt{x^2-1}}{x^{m-1}} + \frac{m-2}{m-1}\cdot\int_x \frac{1}{x^{m-2}\sqrt{x^2-1}},$$

and therefore u may be reduced, m odd, to

$\displaystyle\int_x \frac{1}{x\sqrt{x^2-1}} = \sec^{-1} x$, and m even, to $\displaystyle\int_x \frac{1}{x^2\sqrt{x^2-1}} = \frac{\sqrt{x^2-1}}{x}$.

EXAMPLE. Find $\displaystyle\int_x \frac{1}{x^5\sqrt{x^2-1}}$.

$$\int_x \frac{1}{x^5\sqrt{x^2-1}} = \tfrac{1}{4}\cdot\frac{\sqrt{x^2-1}}{x^4} + \tfrac{3}{4}\cdot\int \frac{1}{x^3\sqrt{x^2-1}},$$

$$\int_x \frac{1}{x^3\sqrt{x^2-1}} = \tfrac{1}{2}\cdot\frac{\sqrt{x^2-1}}{x^2} + \tfrac{1}{2}\cdot\sec^{-1} x;$$

$$\therefore \int_x \frac{1}{x^5\sqrt{x^2-1}} = \tfrac{1}{4}\cdot\frac{\sqrt{x^2-1}}{x^4} + \frac{3}{2\cdot4}\cdot\frac{\sqrt{x^2-1}}{x^2} + \frac{1\cdot3}{2\cdot4}\sec^{-1} x.$$

50. Integrate $\dfrac{du}{dx} = \dfrac{x^m}{\sqrt{2ax-x^2}}$.

$$\int_x \frac{x^m}{\sqrt{2ax-x^2}} = \int_x \frac{-x^{m-1}.(a-x) + ax^{m-1}}{\sqrt{2ax-x^2}}$$

$$= -\int_x \frac{x^{m-1}(a-x)}{\sqrt{2ax-x^2}} + a\cdot\int_x \frac{x^{m-1}}{\sqrt{2ax-x^2}}.$$

Now $\displaystyle\int_x x^{m-1}\frac{a-x}{\sqrt{2ax-x^2}} = x^{m-1}\sqrt{2ax-x^2} - (m-1).\int_x \sqrt{2ax-x^2}.x^{m-2}$

$$= x^{m-1}\sqrt{2ax-x^2} - 2.(m-1)\,a.\int_x \frac{x^{m-1}}{\sqrt{2ax-x^2}} + (m-1)\cdot\int_x \frac{x^m}{\sqrt{2ax-x^2}};$$

therefore, substituting

$$m\int_x \frac{x^m}{\sqrt{2ax-x^2}} = -x^{m-1}\sqrt{2ax-x^2} + (2m-1)\,a\,.\int_x \frac{x^{m-1}}{\sqrt{2ax-x^2}};$$

$$\therefore \int_x \frac{x^m}{\sqrt{2ax-x^2}} = -\frac{x^{m-1}\sqrt{2ax-x^2}}{m} + \frac{2m-1}{m}\cdot a\cdot\int \frac{x^{m-1}}{\sqrt{2ax-x^2}},$$

by which u may be reduced to $\displaystyle\int_x \frac{1}{\sqrt{2ax-x^2}} = V\sin^{-1}\frac{x}{a}$.

The last term

$$= a^m . \frac{(2m-1)(2m-3)(2m-5)\ldots 3.1}{m.(m-1).(m-2)\ldots 2.1} V \sin^{-1}\frac{x}{a}.$$

51. If $u=0$, when $x=0$, and its value be required when $x=2a$. Then, since all the terms vanish when $x=0$; $\therefore C=0$: and when $x=2a$, all the terms of the form $x^{m-1}\sqrt{2ax-x^2}$ vanish; but $V\sin^{-1}\frac{x}{2a}=\pi$;

$$\therefore \int_x \frac{x^m}{\sqrt{2ax-x^2}} \text{ from } x=0 \text{ to } x=2a,$$

$$= \pi . a^m . \frac{1.3.5\ldots(2m-3).(2m-1)}{1.2.3\ldots(m-1) \quad m}.$$

52. Integrate $\frac{du}{dx}=\frac{1}{x^m\sqrt{2ax-x^2}}$. Let $x=\frac{1}{z}$;

$$\therefore \frac{du}{dz}=\frac{z^{m+1}}{\sqrt{2az-1}}\times -\frac{1}{z^2}=\frac{-z^{m-1}}{\sqrt{2az-1}}$$

$$= -\sqrt{\beta}.\frac{z^{m-1}}{\sqrt{z-\beta}}; \text{ if } \beta=\frac{1}{2a},$$

which is easily integrated.

53. Integrate $\frac{du}{dx}=\frac{x^m}{\sqrt{a+bx+cx^2}}$.

$$\int_x \frac{x^m}{\sqrt{a+bx+cx^2}}=\int_x \frac{x^m}{\sqrt{c}\sqrt{\frac{a}{c}+\frac{b}{c}x+x^2}}.$$

Let $x+\frac{b}{2c}=z$; $\therefore x^2+\frac{b}{c}x+\frac{a}{c}=z^2+\frac{a}{c}-\frac{b^2}{4c^2}=z^2\pm\beta^2$;

$$\int_x \frac{x^m}{\sqrt{a+bx+cx^2}}=\frac{1}{\sqrt{c}}\int_z \frac{(z-\alpha)^m}{\sqrt{z^2\pm\beta^2}}; \text{ if } \frac{b}{2c}=\alpha,$$

which may be made to depend upon $\int_z \frac{z^m}{\sqrt{z^2\pm\beta^2}}$.

54. Integrate $\frac{du}{dx}=\frac{1}{x^m\sqrt{a+bx+cx^2}}$.

Let $x=\frac{1}{z}$; $\therefore \frac{du}{dz}=-\frac{1}{z^2}.\frac{du}{dx}=-\frac{z^{m+1}}{z^2\sqrt{az^2+bz+c}}$;

$$\therefore\ u = C - \int_z \frac{z^{m-1}}{\sqrt{az^2+bz+c}};$$

which may be integrated by the preceding method.

55. Integrate $\dfrac{du}{dx} = \dfrac{1}{\sqrt{c-x}\,\sqrt{2ax-x^2}}$.

$$\frac{1}{\sqrt{c-x}\,\sqrt{2ax-x^2}} = \frac{1}{\sqrt{c}}\frac{\left(1-\frac{x}{c}\right)^{-\frac{1}{2}}}{\sqrt{2ax-x^2}}$$

$$= \frac{1}{\sqrt{c}}\cdot\frac{1}{\sqrt{2ax-x^2}}\cdot\left\{1+\frac{1}{2}\cdot\frac{x}{c}+\frac{1\,.\,3}{2\,.\,4}\cdot\frac{x^2}{c^2}+\&c.\right\}$$

and thus u depends upon $\displaystyle\int_x \frac{x^m}{\sqrt{2ax-x^2}}$; this integral is met with in Mechanics.

56. Lastly to prove Bernoulli's series for $\int_x u$.

Since $\displaystyle\int_x u = ux - \int_x x\frac{du}{dx}$,

$$\text{and } \int_x x\frac{du}{dx} = \frac{x^2}{2}\cdot\frac{du}{dx} - \tfrac{1}{2}\cdot\int_x x^2\frac{d^2u}{dx^2},$$

$$\int_x x^2\frac{d^2u}{dx^2} = \frac{x^3}{3}\cdot\frac{d^2u}{dx^2} - \tfrac{1}{3}\cdot\int_x x^3\frac{d^3u}{dx^3},$$

$$\int_x x^3\frac{d^3u}{dx^3} = \frac{x^4}{4}\cdot\frac{d^3u}{dx^3} - \tfrac{1}{4}\cdot\int_x x^4\frac{d^4u}{dx^4};$$

$$\&c.\ldots = \&c.$$

$$\therefore\ \int_x u = ux - \frac{x^2}{1\,.\,2}\cdot\frac{du}{dx} + \frac{x^3}{2\,.\,3}\cdot\frac{d^2u}{dx^2} - \frac{x^4}{2\,.\,3\,.\,4}\cdot\frac{d^3u}{dx^3} + \&c.$$

$$\pm \int_x \frac{d^nu}{dx^n}\cdot\frac{x^n}{1\,.\,2\ldots n}.$$

Ex. Let $u = ax^3 + bx^2 + cx + e$;

$$\therefore\ \frac{du}{dx} = 3ax^2 + 2bx + c,$$

$$\frac{d^2u}{dx^2} = 6ax + 2b,$$

$$\frac{d^3u}{dx^3} = 6a, \quad \text{and } \frac{d^4u}{dx^4} = 0;$$

$$\therefore\ \int_x u = ax^4 + bx^3 + cx^2 + ex - \frac{3ax^4 + 2bx^3 + cx^2}{2} + ax^4 + \frac{bx^3}{3} - \frac{ax^4}{4}$$

$$= \frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + ex.$$

Examples.

(1) $\int_x x^2 . \sqrt{a+bx} = \left\{ \frac{(a+bx)^2}{7} - \frac{2a(a+bx)}{5} + \frac{a^2}{3} \right\} \frac{2(a+bx)^{\frac{3}{2}}}{b^3}.$

(2) $\int_x \frac{x^3}{\sqrt{a+bx}} = \left\{ \frac{(a+bx)^3}{7} - \frac{3}{5} a(a+bx)^2 + a^2.(a+bx) - a^3 \right\} \frac{2\sqrt{a+bx}}{b^4}.$

(3) $\int_x \frac{1}{x\sqrt{a+bx}} = \frac{1}{\sqrt{a}} \log \frac{\sqrt{a+bx} - \sqrt{a}}{\sqrt{a+bx} + \sqrt{a}}.$

(4) $\int_x \frac{1}{x\sqrt{bx-a}} = \frac{2}{\sqrt{a}} \tan^{-1} \sqrt{\frac{bx-a}{a}} = \frac{2}{\sqrt{a}} . \sin^{-1} \sqrt{\frac{bx-a}{bx}}.$

(5) $\int_x \frac{\sqrt{a+bx}}{x^2} = -\frac{\sqrt{a+bx}}{x} + \frac{b}{2\sqrt{a}} \log \left\{ \frac{\sqrt{a+bx} - \sqrt{a}}{\sqrt{a+bx} + \sqrt{a}} \right\}.$

(6) $\int_x \frac{1}{x^2\sqrt{4+3x}} = -\frac{\sqrt{4+3x}}{4x} - \frac{3}{16} \log \frac{\sqrt{4+3x} - 2}{\sqrt{4+3x} + 2}.$

(7) $\int_x \frac{x^2}{(1+x)^{\frac{3}{2}}} = \{x^2 - 4x - 8\} \frac{2}{3\sqrt{1+x}}.$

X (8) $\int_x \frac{1}{x(a+bx)^{\frac{3}{2}}} = \frac{2}{a(a+bx)^{\frac{1}{2}}} + \frac{1}{a\sqrt{a}} . \log \left(\frac{\sqrt{a+bx} - \sqrt{a}}{\sqrt{a+bx} + \sqrt{a}} \right).$

(9) $\int_x \frac{x^2}{(1+x)^{\frac{5}{3}}} = (x^2 - 6x - 9) \frac{3}{4(1+x)^{\frac{2}{3}}}.$

(10) $\int_x \frac{1}{x(1+2x)^{\frac{5}{2}}} = \frac{2+3x}{3} \frac{4}{(1+2x)^{\frac{3}{2}}} + \log \left(\frac{\sqrt{1+2x} - 1}{\sqrt{1+2x} + 1} \right).$

(11) $\int_x \frac{x^3}{(2+x)^{\frac{9}{2}}} = -\left(x^3 + 4x^2 + \frac{32x}{5} + \frac{128}{35} \right) \frac{2}{(2+x)^{\frac{7}{2}}}.$

(12) $$\int_x \frac{x^2}{\sqrt[3]{2+3x}} = (2+3x)^{\frac{2}{3}} \cdot \left\{\frac{4-4x+5x^2}{40}\right\}.$$

(13) $$\int_x \frac{x^3}{(1+x)^{\frac{2}{3}}} = (14x^3 - 18x^2 + 27x - 81)\frac{3\sqrt[3]{1+x}}{4 \cdot 5 \cdot 7}.$$

(14) $$\int_x x^5\sqrt{1+x^2} = \left(x^4 - \frac{4x^2}{5} + \frac{8}{15}\right)\frac{(1+x^2)^{\frac{3}{2}}}{7}.$$

(15) $$\int_x \frac{\sqrt{1+x^2}}{x^6} = -\left\{\frac{1}{x^5} - \frac{2}{3x^3}\right\}\frac{(1+x^2)^{\frac{3}{2}}}{5}.$$

(16) $$\int_x x^3(1+x^2)^{\frac{3}{2}} = \frac{5x^2-2}{35}(1+x^2)^{\frac{5}{2}}.$$

(17) $$\int_x x^3(1+x^2)^{\frac{5}{2}} = \frac{7x^2-2}{63}(1+x^2)^{\frac{7}{2}}.$$

(18) $$\int_x \frac{x^5}{\sqrt{1-x^2}} = -\left\{\frac{x^4}{5} + \frac{4x^2}{15} + \frac{8}{15}\right\}\sqrt{1-x^2}.$$

(19) $$\int_x \frac{x^6}{\sqrt{1-x^2}} = -\left\{\frac{x^5}{6} + \frac{5x^3}{24} + \frac{5x}{16}\right\}\sqrt{1-x^2} + \frac{5}{16}\sin^{-1}x.$$

(20) $$\int_x \frac{1}{x^6\sqrt{1+x^2}} = -\left\{\frac{1}{5x^5} - \frac{4}{15x^3} + \frac{8}{15x}\right\}\sqrt{1+x^2}.$$

(21) $$\int_x \frac{1}{(a+bx^2)^{\frac{5}{2}}} = \left\{\frac{1}{3a(a+bx^2)} + \frac{2}{3a^2}\right\}\frac{x}{\sqrt{a+bx^2}}.$$

(22) $$\int_x (1-2x^2)^{\frac{3}{2}} = \frac{3}{8}\left\{\frac{\sin^{-1}.x\sqrt{2}}{\sqrt{2}} + x\sqrt{1-2x^2}\right\} + \tfrac{1}{4}x(1-2x^2)^{\frac{3}{2}}.$$

(23) $$\int_x \frac{x^3}{(1+x^2)^{\frac{3}{2}}} = (x^2+2)\frac{1}{\sqrt{1+x^2}}.$$

(24) $$\int_x \frac{x^4}{(1+x^2)^{\frac{3}{2}}} = \frac{x^3+3x}{2}\frac{1}{\sqrt{1+x^2}} - \frac{3}{2}\log(x+\sqrt{1+x^2}).$$

(25) $$\int_x \frac{x^3}{(1+x^2)^{\frac{5}{2}}} = -\left(x^2+\frac{2}{3}\right)\frac{1}{(1+x^2)^{\frac{3}{2}}}.$$

(26) $$\int_x \frac{x^5}{(1+x^2)^{\frac{5}{2}}} = \left(x^4+4x^2+\frac{8}{3}\right)\frac{1}{(1+x^2)^{\frac{3}{2}}}.$$

(27) $$\int_x \frac{x^2}{(1+x^2)^{\frac{9}{2}}} = \left(\frac{8x^7}{105} + \frac{4x^5}{15} + \frac{x^3}{3}\right)\frac{1}{(1+x^2)^{\frac{7}{2}}}.$$

(28) $$\int_x \frac{1}{(1-x)\sqrt{1-2x}} = -\tan^{-1}.\left(\frac{\sqrt{1-2x}}{x}\right).$$

(29) $$\int_x \frac{1}{(2+x)\sqrt{1+x}} = 2\tan^{-1}\sqrt{1+x}.$$

(30) $$\int_x \frac{1}{\sqrt{1+x+x^2}} = \log(2x+1+2\sqrt{1+x+x^2}).$$

(31) $$\int_x \frac{1}{\sqrt{1+x-x^2}} = \sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right).$$

(32) $$\int_x \frac{1}{(1+x)\sqrt{1-x}} = \frac{1}{\sqrt{2}}.\log\left(\frac{x-3+2\sqrt{2}\sqrt{1-x}}{1+x}\right).$$

(33) $$\int_x \frac{1}{x\sqrt{1+x+x^2}} = \log\left(\frac{2+x-2\sqrt{1+x+x^2}}{x}\right).$$

(34) $$\int_x \frac{1}{x\sqrt{x^2+x-1}} = \sin^{-1}.\left(\frac{x-2}{x\sqrt{5}}\right).$$

(35) $$\int_x \frac{1}{(1+x)\sqrt{1+x^2}} = \frac{1}{\sqrt{2}}.\log\left(\frac{1-x-\sqrt{2}\sqrt{1+x^2}}{1+x}\right).$$

(36) $$\int_x \frac{1}{(1+x^2)\sqrt{1-x^2}} = \frac{1}{\sqrt{2}}\cos^{-1}.\sqrt{\frac{1-x^2}{1+x^2}}.$$

(37) $$\int_x \frac{1}{(1-x^2)\sqrt{1+x^2}} = \frac{1}{\sqrt{2}}\log\left(\frac{\sqrt{1+x^2}+x\sqrt{2}}{\sqrt{1-x^2}}\right).$$

(38) $$\int_x \frac{1}{(1+x)\sqrt{1+x-x^2}} = \tan^{-1}.\left(\frac{1+3x}{2\sqrt{1+x-x^2}}\right).$$

(39) $$\int_x \frac{x}{\sqrt{a^4+x^4}} = \log\sqrt{x^2+\sqrt{a^4+x^4}}.$$

(40) $$\int_x \frac{1}{(1+x)\sqrt{1-x-x^2}} = \log\left(\frac{x+3-2\sqrt{1-x-x^2}}{1+x}\right).$$

(41) $$\int_x \frac{1}{\sqrt{1+2x-x^2}} = \cos^{-1}.\left(\frac{1-x}{\sqrt{2}}\right).$$

(42) $$\int_x \frac{x}{\sqrt{a^4-x^4}} = \frac{1}{2}.\sin^{-1}.\left(\frac{x^2}{a^2}\right).$$

(43) $$\int_x \frac{x}{\sqrt{(x^2-a^2)(b^2-x^2)}} = \sin^{-1}.\sqrt{\frac{x^2-a^2}{b^2-a^2}}.$$

(44) $$\int_x \frac{1}{(a+bx^2)^{\frac{3}{2}}} = \frac{x}{a\sqrt{a+bx^2}}.$$

(45) $$\int_x \sqrt{\frac{1+x}{1-x}} = \sin^{-1}x - \sqrt{1-x^2}.$$

(46) $$\int_x \frac{1}{x}\sqrt{\frac{1+x}{1-x}} = \sin^{-1}x + \log\left(\frac{x}{1+\sqrt{1-x^2}}\right).$$

(47) $$\int_x x\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\sin^{-1}x - \frac{1}{2}(2+x)\sqrt{1-x^2}.$$

(48) $$\int_x \frac{\sqrt{x}}{\sqrt{a^3-x^3}} = \frac{2}{3}\,.\tan^{-1}\sqrt{\frac{x^3}{a^3-x^3}}.$$

(49) $$\int_x \frac{1}{(2ax+x^2)^{\frac{3}{2}}} = -\frac{x+a}{a^2\sqrt{2ax+x^2}}.$$

(50) $$\int_x \frac{1}{(2ax+x^2)^{\frac{5}{2}}} = -\left\{\frac{1}{3(2ax+x^2)} - \frac{2}{3a^2}\right\}\frac{x+a}{a^2\sqrt{2ax+x^2}}.$$

(51) $$\int_x \frac{1}{(1+x+x^2)^{\frac{3}{2}}} = \frac{2\,.(2x+1)}{3\sqrt{1+x+x^2}}.$$

(52) $$\int_x \frac{1}{(1+x+x^2)^{\frac{5}{2}}} = \left\{\frac{1}{1+x+x^2} + \frac{8}{3}\right\}\frac{2(2x+1)}{9\sqrt{1+x+x^2}}.$$

(53) $$\int_x \frac{1}{(1+x)^2\sqrt{x}} = \frac{\sqrt{x}}{(1+x)} + \tan^{-1}(\sqrt{x}).$$

(54) $$\int_x \frac{x\sqrt{x}}{1+x} = \left(\frac{x}{3}-1\right)2\sqrt{x} + 2\tan^{-1}\sqrt{x}.$$

(55) $$\int_x \frac{x\sqrt{x}}{1+x^2} = 2\sqrt{x} + \frac{1}{\sqrt{2}}\left\{\log\left(\frac{x+1+\sqrt{2x}}{\sqrt{1+x^2}}\right) - \tan^{-1}\frac{\sqrt{2x}}{1-x}\right\}.$$

(56) $$\int_x \frac{1}{x^2\sqrt{2ax-x^2}} = -\frac{\sqrt{2ax-x^2}}{3a^2x^2}(a+x).$$

(57) $$\int_x \frac{x^2}{\sqrt{a+bx+cx^2}} = \sqrt{a+bx+cx^2}\left\{\frac{2cx-3b}{4c^2}\right\}$$
$$+\frac{4ac-3b^2}{8c^2\sqrt{c}}\log\left\{\frac{2cx+b+2\sqrt{c}\sqrt{a+bx+cx^2}}{2c}\right\}$$

(58) Rationalize the integrals

$$(1)\quad \int_x \frac{x^3 + 2x^{\frac{2}{3}} + x^{\frac{1}{3}}}{x + 3x^{\frac{1}{4}}}, \qquad (2)\quad \int_x \frac{x^4}{x^5 + \sqrt[4]{(1+x)^3}},$$

in (1) make $x = z^{12}$, and in (2) make $(1+x) = z^4$.

(59) $$\int_x x^3 \sqrt{\frac{1+x^2}{1-x^2}} = \frac{1}{2}\tan^{-1}\sqrt{\frac{1+x^2}{1-x^2}}$$

$$-\frac{1}{4}(2+x^2)\sqrt{1-x^4}.$$

(60) $$\int_x \frac{1}{x}\sqrt{\frac{a^2-c^2x^2}{a^2-x^2}} = \log\sqrt{\left(\frac{y-1}{y+1}\right)\left(\frac{y+c}{y-c}\right)^c}$$

$$\text{where } y = \sqrt{\frac{a^2-c^2x^2}{a^2-x^2}}.$$

(61) $$\int_x \frac{x^2+1}{x^2-1}\,\frac{1}{\sqrt{1-ax^2+x^4}} = \frac{1}{\sqrt{a-2}}\cos^{-1}.\left(\frac{x\sqrt{a-2}}{x^2-1}\right)$$

$$\text{make } x - \frac{1}{x} = \frac{1}{z}.$$

(62) $$\int_x \frac{1-x^2}{1+x^2}\cdot\frac{1}{\sqrt{1+x^4}} = \frac{1}{\sqrt{2}}.\sin^{-1}\frac{x\sqrt{2}}{1+x^2}.$$

CHAPTER IV.

Integrals of Logarithmic and Exponential Functions.

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57. THESE functions are of the form $X(\log x)^n$. $X.a^x$, where X is a function of x.

58. Integrate $\int_x X.(\log x)^n$.

Let $\int_x X = P$, $\int_x P.\frac{1}{x} = Q$, and $\int_x Q.\frac{1}{x} = R$.

Then $\int_x X(\log x)^n = P(\log x)^n - n.\int_x P.(\log x)^{n-1}.\frac{1}{x}$,

and $\int_x \frac{P}{x}.(\log x)^{n-1} = Q(\log x)^{n-1} - (n-1).\int_x Q.(\log x)^{n-2}.\frac{1}{x}$,

$$\int_x \frac{Q}{x}.(\log x)^{n-2} = R(\log x)^{n-2} - (n-2).\int_x R.(\log x)^{n-3}.\frac{1}{x};$$

$$\therefore \int_x X(\log x)^n = P(\log x)^n - n.Q(\log x)^{n-1} + n.(n-1).R.(\log x)^{n-2} - \&c.$$

59. $\int_x x^m(\log x)^n$.

$$\int_x x^m(\log x)^n = \frac{x^{m+1}(\log x)^n}{m+1} - \frac{n}{m+1}.\int_x x^{m+1}.(\log x)^{n-1}.\frac{1}{x}$$

$$= \frac{x^{m+1}(\log x)^n}{m+1} - \frac{n}{m+1}.\int_x x^m.(\log x)^{n-1},$$

$$\int_x x^m(\log x)^{n-1} = \frac{x^{m+1}(\log x)^{n-1}}{m+1} - \frac{n-1}{m+1}.\int_x x^m.(\log x)^{n-2},$$

and in this manner may the integral be reduced to

$$\int_x x^m = \frac{x^{m+1}}{m+1}, \text{ if } n \text{ be a whole number,}$$

and $\therefore \int_x x^m.(\log x)^n = \frac{x^{m+1}}{m+1}\{(\log x)^n - \frac{n}{m+1}(\log x)^{n-1}$

$$+ \frac{n.(n-1)}{(m+1)^2}(\log x)^{n-2} - \&c.\} \pm \frac{n.(n-1).(n-2)\ldots2.1}{(m+1)^{n+1}}x^{m+1}.$$

Every term of the integral vanishes both when $x=0$ and $x=1$, except the last, which vanishes only when $x=0$;

$$\therefore \int_x x^m (\log x)^n, \quad \left.\begin{matrix}\text{from } x=0\\ \text{to } x=1\end{matrix}\right\} = \pm \frac{1.2.3\ldots(n-1).n}{(m+1)^{n+1}}.$$

60. Integrate $\int_x \frac{X}{(\log x)^n}$, n a whole number.

$$\int_x \frac{X}{(\log x)^n} = \int_x \frac{X.x.\frac{d.\log x}{dx}}{(\log x)^n}. \quad \text{Since } x\frac{d.\log x}{dx} = x\frac{1}{x} = 1$$

$$= \frac{-X.x}{(n-1)(\log x)^{n-1}} + \frac{1}{n-1}.\int_x \frac{\frac{d.(Xx)}{dx}}{(\log x)^{n-1}}.$$

Let $\frac{d.(Xx)}{dx} = P$;

$$\therefore \int_x \frac{X}{(\log x)^n} = \frac{-Xx}{(n-1)(\log x)^{n-1}} + \frac{1}{n-1}.\int_x \frac{P}{(\log x)^{n-1}};$$

and $\int_x \frac{P}{(\log x)^{n-1}} = \frac{-P.x}{(n-2)(\log x)^{n-2}} + \frac{1}{n-2}.\int_x \frac{Q}{(\log x)^{n-2}}$,

where $Q = \frac{d.(P.x)}{dx}$;

$$\therefore \int_x \frac{X}{(\log x)^n} = \frac{-Xx}{(n-1)(\log x)^{n-1}} - \frac{Px}{(n-1)(n-2).(\log x)^{n-2}}$$

$$- \frac{Q.x}{(n-1)(n-2)(n-3)(\log x)^{n-3}} - \&c.$$

in this manner the integral may be reduced to $\int_x \frac{X_1}{(\log x)}$, which cannot be integrated except by a series.

61. Find $\int_x \frac{x^m}{(\log x)^2}$.

$$\int_x \frac{x^m}{(\log x)^2} = \int_x \frac{x^{m+1}.\frac{1}{x}}{(\log x)^2} = \frac{-x^{m+1}}{\log x} + (m+1).\int_x \frac{x^m}{\log x}.$$

Let $\log x = z$; $\therefore x = e^z$, and $x^m = e^{mz}$;

$$\therefore \int_x \frac{x^m}{\log x} = \int_z \frac{e^{mz}}{z}.\frac{dx}{dz} = \int_z \frac{e^{mz}}{z}e^z = \int_z \frac{e^{(m+1)z}}{z}$$

$$= \int_z \{1 + (m+1)z + \frac{(m+1)^2z^2}{1.2} + \frac{(m+1)^3z^3}{2.3} + \&c.\}.\frac{1}{z}.$$

$$=\log z+(m+1)\,.\,z+\frac{(m+1)^2z^2}{1\,.\,2^2}+\frac{(m+1)^3z^3}{2\,.\,3^2}+\&c.$$

$$=\log(\log x)+(m+1)\log x+\frac{(m+1)^2(\log x)^2}{1\,.\,2^2}+\frac{(m+1)^3(\log x)^3}{2\,.\,3^2}+\&c.$$

COR. If $m=0$, we have

$$\int_x\frac{1}{\log x}=\log(\log x)+(\log x)+\frac{(\log x)^2}{1\,.\,2^2}+\&c.$$

62. Integrate $\int_x a^x\,.\,X$, X being a function of x.

Since $\dfrac{d\,.\,a^x}{dx}=a^x\log a=Aa^x$; $\therefore\ \int_x a^x=\dfrac{a^x}{A}$;

$$\therefore\ \int_x a^x\,.\,X=\frac{Xa^x}{A}-\frac{1}{A}\,.\int\frac{dX}{dx}\,.\,a^x.$$

$$\text{Let } \frac{dX}{dx}=P,\quad \frac{dP}{dx}=Q,\quad \&c.;$$

$$\therefore\ \int_x a^x\,.\,X=\frac{Xa^x}{A}-\frac{1}{A}\,.\int_x Pa^x,$$

$$\int_x a^x\,.\,P=\frac{Pa^x}{A}-\frac{1}{A}\,.\int Qa^x;$$

$$\therefore\ \int_x a^x\,.\,X=a^x\,.\left\{\frac{X}{A}-\frac{P}{A^2}+\frac{Q}{A^3}-\&c.\right\}.$$

Ex. 1. Let $\int_x x^m\,.\,a^x$ be required.

$$\int_x x^m\,.\,a^x=\frac{x^m\,.\,a^x}{A}-\frac{m}{A}\,.\int_x x^{m-1}\,.\,a^x,$$

$$\int_x x^{m-1}\,.\,a^x=\frac{x^{m-1}\,.\,a^x}{A}-\frac{m-1}{A}\,.\int_x x^{m-2}\,.\,a^x;$$

$$\therefore\ \int_x x^m\,.\,a^x=a^x\,.\left\{\frac{x^m}{A}-\frac{mx^{m-1}}{A^2}+\frac{m\,.\,(m-1)\,x^{m-2}}{A^3}-\&c.\right\}.$$

Ex. 2. $\displaystyle\int_x\frac{a^x}{x^n}=\frac{-a^x}{(n-1)\,x^{n-1}}+\frac{A}{n-1}\,.\int_x\frac{a^x}{x^{n-1}}$;

$$\therefore\ \int_x\frac{a^x}{x^{n-1}}=\frac{-a^x}{(n-2)\,x^{n-2}}+\frac{A}{n-2}\,.\int_x\frac{a^x}{x^{n-2}},$$

$$\therefore\ \int_x\frac{a^x}{x^n}=\frac{-a^x}{(n-1)\,.\,x^{n-1}}-\frac{Aa^x}{(n-1)\,.\,(n-2)\,.\,x^{n-2}}$$

$$\frac{A^2 a^x}{(n-1).(n-2).(n-3).x^{n-3}} - \&c.$$

$$+ \frac{A^{n-1}}{(n-1).(n-2)\ldots 1} \cdot \int_x \frac{a^x}{x}.$$

Also $\displaystyle\int_x \frac{a^x}{x} = \int_x \frac{\left(1 + Ax + \frac{A^2x^2}{1.2} + \frac{A^3x^3}{2.3} + \&c.\right)}{x}$

$$= \int_x \left(\frac{1}{x} + A + \frac{A^2 x}{1.2} + \frac{A^3 x^2}{2.3} + \&c.\right)$$

$$= \log x + Ax + \frac{A^2x^2}{1.2^2} + \frac{A^3x^3}{2.3^2} + \frac{A^4x^4}{2.3.4^2} + \&c.$$

Ex. 3. Find $\displaystyle\int_x \frac{1}{x}.\log(a+bx)$.

$$\log(a+bx) = \log a\left(1 + \frac{b}{a}x\right) = \log a + \log\left(1 + \frac{b}{a}x\right)$$

$$= \log a + \left(\frac{b}{a}x - \frac{b^2x^2}{2a^2} + \frac{b^3x^3}{3a^3} - \frac{b^4x^4}{4a^4} + \&c.\right);$$

$$\therefore \int_x \frac{1}{x}\log(a+bx) = \log x.\log a$$

$$+ \left(\frac{b}{a}x - \frac{b^2x^2}{2^2a^2} + \frac{b^3x^3}{3^2a^3} - \frac{b^4x^4}{4^2x^4} + \&c.\right).$$

Ex. 4. Find $\int_x x^{nx}$.

$$x^{nx} = 1 + nx\log x + \frac{(nx\log x)^2}{1.2} + \frac{(nx\log x)^3}{1.2.3} + \&c.;$$

$$\therefore \int_x x^{nx} = x + n.\int_x x\log x + \frac{n^2}{1.2}.\int_x x^2(\log x)^2$$

$$+ \frac{n^3}{2.3}.\int_x x^3(\log x)^3 + \&c.$$

Hence, the integration depends upon $\int_x x^m(\log x)^m$, Art. 59;

$$\therefore \int_x x\log x = \frac{x^2}{2}.\left(\log x - \tfrac{1}{2}\right),$$

$$\int_x x^2(\log x)^2 = \frac{x^3}{3}\left\{(\log x)^2 - \tfrac{2}{3}\log x + \frac{2.1}{3^2}\right\},$$

$$\int_x (x\log x)^3 = \frac{x^4}{4}\left\{(\log x)^3 - \tfrac{3}{4}(\log x)^2 + \frac{3.2}{4^2}(\log x) - \frac{3.2}{4^3}\right\},$$

&c.

and arranging the terms according to the powers of $\log x$,

$$\int_x x^{nx} = x - \frac{nx^2}{2^2} + \frac{n^2x^3}{3^3} - \frac{n^3x^4}{4^4} + \frac{n^4x^5}{5^5} - \&c.$$

$$+ \left(\frac{x^2}{2} - \frac{nx^3}{3^2} + \frac{n^2x^4}{4^3} - \&c.\right) n \log x$$

$$+ \left(\frac{x^3}{3} - \frac{nx^4}{4^2} + \frac{n^2x^5}{5^3} - \&c.\right) \frac{n^2 (\log x)^2}{1.2}$$

$$+ \left(\frac{x^4}{4} - \frac{nx^5}{5^2} + \frac{n^2x^6}{6^3} - \&c.\right) \frac{n^3(\log x)^3}{1.2.3}$$

$$+ \&c.$$

COR. Since $x^m (\log x)^n = 0$; both when $x = 0$, and $x = 1$;

$$\therefore \int_x x^{nx}, \quad \left.\begin{matrix}\text{from } x = 0 \\ \text{to } x = 1\end{matrix}\right\} = 1 - \frac{n}{2^2} + \frac{n^2}{3^3} - \frac{n^3}{4^4} + \frac{n^4}{5^5} - \&c.,$$

and if $n = 1$, $\int_x x^x$ between the same values

$$= \frac{1}{1} - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \frac{1}{5^5} - \&c.$$

This last integral gives the area of a curve, defined by $y = x^x$, included between two ordinates, each $= 1$, one drawn through the origin, and the other at a distance $= 1$ from it.

Ex. 5. $\displaystyle\int_x e^x . \frac{x}{(1+x)^2}$,

$$e^x . \frac{x}{(1+x)^2} = e^x . \left\{\frac{1}{1+x} - \frac{1}{(1+x)^2}\right\} = \frac{d}{dx} . \left(\frac{e^x}{1+x}\right);$$

$$\therefore \int_x e^x . \frac{x}{(1+x)^2} = \frac{e^x}{1+x}.$$

63. Find $\int_t e^{-t^2}$ between $t = -\infty$ and $t = \infty$.

Now by Art. 46, between $x = 0$ and $x = 1$, we have

$$\left(\int_x \frac{x^{2n}}{\sqrt{1-x^2}}\right)\left(\int_x \frac{x^{2n+1}}{\sqrt{1-x^2}}\right) = \frac{1}{2n+1} . \frac{\pi}{2}.$$

Let $x = e^{-qt^2}$; $\therefore \dfrac{dx}{dt} = -2qt . e^{-qt^2}$;

$$\therefore 4q^2 \left\{ \int_t \frac{te^{-q(2n+1)t^2}}{\sqrt{1-e^{-2qt^2}}} \times \int_t \frac{te^{-qt^2} . e^{-qt^2(2n+1)}}{\sqrt{1-e^{-2qt^2}}} \right\} = \frac{1}{2n+1} . \frac{\pi}{2},$$

the limits of t being $-\infty$ and 0; now let $q=\frac{1}{2n+1}$;

$$\therefore\ 2\left\{\int_t \frac{te^{-t^2}}{\sqrt{\frac{1-e^{-2qt^2}}{2q}}}\right\}\left\{\int_t \frac{te^{-t^2(1+q)}}{\sqrt{\frac{1-e^{-2qt^2}}{2q}}}\right\}=\frac{\pi}{2}.$$

Now making $q=0$, in which case $\sqrt{\frac{1-e^{-2qt^2}}{2q}}=t$;

$$\therefore\ 2(\textstyle\int_t e^{-t^2})^2=\frac{\pi}{2};\quad \therefore\ \int_t e^{-t^2}=\frac{1}{2}\sqrt{\pi};$$

but as the value of the integral is the same for t positive as t negative; $\therefore$ the integral, from $t=0$ to $t=\infty$, $=\frac{1}{2}\sqrt{\pi}$;

$$\therefore\ \text{from } t=-\infty \text{ to } t=+\infty;\quad \int_t e^{-t^2}=\sqrt{\pi}.$$

Examples.

(1) $\int_x x^4(\log x)=\frac{x^5}{5}\left(\log x-\frac{1}{5}\right).$

(2) $\int_x x^2(\log x)^2=\frac{x^3}{3}.\left\{(\log x)^2-\frac{2}{3}\log x+\frac{2}{9}\right\}.$

(3) $\int_x x^3(\log x)^3=\frac{x^4}{4}\left\{(\log x)^3-\frac{3}{4}(\log x)^2+\frac{3.2}{4^2}\log x-\frac{3.2}{4^3}\right\}.$

(4) $\int_x \frac{1}{x}\log x=\frac{1}{2}(\log x)^2;\quad \int_x \frac{1}{x\log x}=\log.\log x.$

(5) $\int_x \frac{x^4}{(\log x)^3}=-\frac{x^5}{2(\log x)^2}-\frac{5x^5}{2.\log x}+\frac{25}{2}.\int_x \frac{x^4}{\log x}.$

(6) $\int_x \frac{x^3}{\sqrt{\log x}}=\frac{x^4}{4\sqrt{\log x}}.\{1+\frac{1}{8\log x}+\frac{1.3}{(8.\log x)^2}$
$+\frac{1.3.5}{(8.\log x)^3}+\&c.\}.$

(7) $\int_x a^x.x^3=a^x\left\{\frac{x^3}{A}-\frac{3x^2}{A^2}+\frac{6x}{A^3}-\frac{6}{A^4}\right\}.$

(8) $\int_x e^x x^4=e^x\{x^4-4x^3+12x^2-24x+24\}.$

(9) $\int_x e^{-x}x^3=-e^{-x}\{x^3+3x^2+6x+6\}.$

(10) $\int_x xe^{\sqrt{x}} = 2e^{\sqrt{x}}\{x^{\frac{3}{2}} - 3x + 6\sqrt{x} - 6\}$.

(11) $\displaystyle\int_x \frac{a^x}{x^4} = -a^x\left\{\frac{1}{3x^3} + \frac{A}{2.3x^2} + \frac{A^2}{2.3.x}\right\} + \frac{A^3}{2.3}.\int_x \frac{a^x}{x}$.

(12) $\displaystyle\int_x \frac{a^x}{\sqrt{x}} = \frac{a^x}{A\sqrt{x}}\{1 + \frac{1}{2xA} + \frac{3}{(2xA)^2} + \frac{3.5}{(2xA)^3} + \&c.\}$,

also $\displaystyle = \frac{a^x}{A\sqrt{x}}.\left\{\frac{2xA}{1} - \frac{(2xA)^2}{1.3} + \frac{(2xA)^3}{3.5} - \frac{(2xA)^4}{3.5.7} + \&c.\right\}$.

(13) $\displaystyle\int_x \frac{\log x}{(1+x)^2} = \frac{x}{x+1}\log x - \log(1+x)$.

(14) $\displaystyle\int \frac{x}{\sqrt{1+x^2}}\log x = \sqrt{1+x^2}.\log\left(\frac{x}{e}\right) - \log\left(\frac{x}{1+\sqrt{1+x^2}}\right)$.

(15) $\displaystyle\int_x \frac{1}{e^x+1} = \log\left(\frac{e^x}{e^x+1}\right)$.

(16) $\displaystyle\int_x e^x.\frac{x^2+1}{(x+1)^2} = e^x.\left(\frac{x-1}{x+1}\right)$.

(17) $\displaystyle\int_x \frac{e^x.(2-x^2)}{(1-x)\sqrt{1-x^2}} = e^x.\sqrt{\frac{1+x}{1-x}}$.

(18) $\displaystyle\int_x \frac{xe^x}{(e^x-1)^3} = \log\frac{e^x}{e^x-1} - \frac{1}{e^x-1}\left\{1 + \frac{x}{2(e^x-1)}\right\}$.

(19) $\displaystyle\int_x x^{nx}.x^m\left\{\begin{matrix}\text{from } x=0\\ \text{to } x=1\end{matrix}\right\} = \frac{1}{m+1} - \frac{n}{(m+2)^2}$

$\displaystyle + \frac{n^2}{(m+3)^3} - \frac{n^3}{(m+4)^4} + \&c.$

(20) $\displaystyle\int x^n e^{-x}\left\{\begin{matrix}\text{from } x=0\\ \text{to } x=\infty\end{matrix}\right\} = n(n-1)(n-2)\ldots 2.1$.

CHAPTER V.

Circular Functions.

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64. THESE are of the form $\sin^n\theta$, $\cos^n\theta$, $(\sin\theta)^m$. $(\cos\theta)^n$, $\frac{1}{(\sin\theta)^n}$, and $X.\sin^{-1}x$, where X is a function of x; these may be integrated by parts, and be reduced either to known or more simple functions such as,

$$\sin\theta,\ \cos\theta,\ \frac{1}{\cos^2\theta},\ \tan\theta,\ \cot\theta,\ \frac{1}{\sin\theta},\ \frac{1}{\cos\theta},\ \text{and}\ \frac{1}{\cos\theta\sin\theta}.$$

Also (1) $\int_\theta \sin\theta = -\cos\theta$. (2) $\int_\theta \cos\theta = \sin\theta$.

(3) $\int_\theta \frac{1}{\cos^2\theta} = \tan\theta$. (4) $\int_\theta \frac{1}{\sin^2\theta} = -\cot\theta$.

(5) $\int_\theta \tan\theta = \int_\theta \frac{\sin\theta}{\cos\theta} = -\log\cos\theta$.

(6) $\int_\theta \cot\theta = \int_\theta \frac{\cos\theta}{\sin\theta} = \log\sin\theta$.

65. Integrate $\frac{1}{\sin\theta}$, $\frac{1}{\cos\theta}$, and $\frac{1}{\sin\theta\cos\theta}$.

(1) $$\int_\theta \frac{1}{\sin\theta} = \int_\theta \frac{\sin\theta}{1-\cos^2\theta} = \tfrac{1}{2}\int_\theta \left(\frac{\sin\theta}{1-\cos\theta} + \frac{\sin\theta}{1+\cos\theta}\right)$$

$$= \tfrac{1}{2}\log\left(\frac{1-\cos\theta}{1+\cos\theta}\right) = \tfrac{1}{2}\log\frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}} = \log\left(\tan\frac{\theta}{2}\right).$$

(2) $$\int_\theta \frac{1}{\cos\theta} = \int_\theta \frac{\cos\theta}{1-\sin^2\theta} = \tfrac{1}{2}\int_\theta \left\{\frac{\cos\theta}{1+\sin\theta} + \frac{\cos\theta}{1-\sin\theta}\right\}$$

$$= \tfrac{1}{2}.\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right) = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right).$$

(3) $$\int_\theta \frac{1}{\sin\theta\cos\theta} = \int_\theta \frac{\sin^2\theta + \cos^2\theta}{\sin\theta.\cos\theta} = \int_\theta \frac{\sin\theta}{\cos\theta} + \int_\theta \frac{\cos\theta}{\sin\theta}$$

$$= -\log\cos\theta + \log\sin\theta = \log(\tan\theta).$$

66. Find $\int_x X \,.\, \sin^{-1}x$, where X is a function of x. Make $\int_x X = P$, and then integrating by parts,

$$\int_x X \,.\, \sin^{-1}x = P\sin^{-1}x - \int_x \frac{P}{\sqrt{1-x^2}},$$

and $\int_x \frac{P}{\sqrt{1-x^2}}$ has been integrated.

Ex. $\int_x \frac{x}{\sqrt{1-x^2}}\sin^{-1}x$. Here $P = -\sqrt{1-x^2}$;

$$\therefore \int_x \frac{x\sin^{-1}x}{\sqrt{1-x^2}} = -\sqrt{1-x^2}\,.\,\sin^{-1}x + \int_x \sqrt{1-x^2}\,\frac{1}{\sqrt{1-x^2}}$$

$$= -\sqrt{1-x^2}\,.\,\sin^{-1}x + x.$$

Similarly may $\int_x X\cos^{-1}x$ be integrated.

67. To integrate $\frac{du}{dx} = X\tan^{-1}x$.

Let $\int_x X = P$;

$$\therefore \int_x X\tan^{-1}x = P\tan^{-1}x - \int_x \frac{P}{1+x^2}.$$

68. Integrate $\frac{du}{d\theta} = \sin^n\theta$.

Integrating by parts, since $\sin^n\theta = \sin^{n-1}\theta\,.\,\sin\theta$;

$$\therefore \int_\theta \sin^n\theta = \int \sin^{n-1}\theta\,.\,\sin\theta$$

$$= -\sin^{n-1}\theta\,.\,\cos\theta + (n-1)\,.\int_\theta \sin^{n-2}\theta\cos^2\theta;$$

and putting $1-\sin^2\theta$ for $\cos^2\theta$

$$= -\sin^{n-1}\theta\cos\theta + (n-1)\,.\int_\theta \sin^{n-2}\theta - (n-1)\int_\theta \sin^n\theta;$$

$$\therefore \int_\theta\,.\,\sin^n\theta = -\frac{\sin^{n-1}\theta\,.\,\cos\theta}{n} + \frac{n-1}{n}\int_\theta \sin^{n-2}\theta,$$

a formula by which $\int_\theta \sin^n\theta$ may be reduced to $-\cos\theta$, or θ, according as n is odd or even.

Suppose n to be even or $=2m$, to find the value of $\int_\theta (\sin\theta)^{2m}$ between $\theta = 0$, and $\theta = \frac{\pi}{2}$.

Let $\int_\theta (\sin\theta)^{2m} = P_{2m}$; $\sin^{2m-1}\theta\cos\theta = Q_{2m-1}$;

$$\therefore P_{2m} = -\frac{1}{2m}\,.\,Q_{2m-1} + \frac{2m-1}{2m}P_{2m-2}.$$

But $Q_{2m-1}=0$ both for $\theta=0$ and $\theta=\frac{\pi}{2}$;

$$\therefore\ P_{2m}=\frac{2m-1}{2m}P_{2m-2};\quad \therefore\ P_{2m-2}=\frac{2m-3}{2m-2}.P_{2m-4};$$

$$\therefore\ P_{2m}=\frac{(2m-1).(2m-3)}{2m.(2m-2)}.P_{2m-4};\ \text{hence}\ \therefore$$

$$=\frac{(2m-1).(2m-3)\ldots 3.1}{2m.(2m-2).4.2}.P_0$$

$$=\frac{1.3.5\ldots(2m-1)}{2.4.6\ldots 2m}.\frac{\pi}{2}.$$

69. Integrate $\frac{du}{d\theta}=\cos^n\theta$.

$$\int_\theta \cos^n\theta=\int_\theta \cos^{n-1}\theta.\cos\theta$$

$$=+\cos^{n-1}\theta\sin\theta+(n-1).\int_\theta \cos^{n-2}\theta\sin^2\theta$$

$$=\cos^{n-1}\theta\sin\theta+(n-1)\int_\theta\cos^{n-2}\theta-(n-1)\int_\theta\cos^n\theta$$

$$=\frac{\cos^{n-1}\theta\sin\theta}{n}+\frac{n-1}{n}\int_\theta\cos^{n-2}\theta,$$

a formula by which $\int_\theta\cos^n\theta$ may be reduced to $\sin\theta$ or θ, according as n is odd or even.

70. Let $\frac{du}{d\theta}=\frac{1}{(\sin\theta)^n}$. Since $\sin^2\theta+\cos^2\theta=1$;

$$\therefore\ u=\int_\theta\frac{\sin^2\theta+\cos^2\theta}{(\sin\theta)^n}=\int\frac{1}{(\sin\theta)^{n-2}}+\int_\theta\frac{\cos^2\theta}{(\sin\theta)^n},$$

and $$\int_\theta\frac{\cos^2\theta}{(\sin\theta)^n}=-\frac{\cos\theta}{(n-1)(\sin\theta)^{n-1}}-\frac{1}{n-1}\int_\theta\frac{\sin\theta}{(\sin\theta)^{n-1}};$$

$$\therefore\ u=-\frac{\cos\theta}{(n-1)(\sin\theta)^{n-1}}+\left(1-\frac{1}{n-1}\right)\int_\theta\frac{1}{(\sin\theta)^{n-2}}$$

$$=-\frac{\cos\theta}{(n-1)(\sin\theta)^{n-1}}+\frac{n-2}{n-1}.\int_\theta\frac{1}{(\sin\theta)^{n-2}},$$

a formula by which n may be diminished.

71. If $\frac{du}{d\theta}=\frac{1}{(\cos\theta)^n}$, then, as in last article,

$$u=\int_\theta\frac{1}{(\cos\theta)^{n-2}}+\int_\theta\frac{\sin^2\theta}{(\cos\theta)^n},$$

and $\int_\theta \frac{\sin^2\theta}{(\cos\theta)^n} = \frac{\sin\theta}{(n-1).(\cos\theta)^{n-1}} - \frac{1}{n-1}\int_\theta \frac{\cos\theta}{(\cos\theta)^{n-1}}$;

$$\therefore\ u = \frac{\sin\theta}{(n-1)(\cos\theta)^{n-1}} + \frac{n-2}{n-1}\int_\theta \frac{1}{(\cos\theta)^{n-2}}.$$

72. Let $\frac{du}{d\theta} = (\sin\theta)^m(\cos\theta)^n$, m and n both integers,

$(\sin\theta)^m(\cos\theta)^n = (\sin\theta)^m\cos\theta\,(\cos\theta)^{n-1}$;

$$\therefore\ \int_\theta(\sin\theta)^m(\cos\theta)^n = \frac{(\sin\theta)^{m+1}(\cos\theta)^{n-1}}{m+1} + \frac{n-1}{m+1}\int_\theta(\sin\theta)^{m+2}(\cos\theta)^{n-2}$$

$$= \frac{(\sin\theta)^{m+1}(\cos\theta)^{n-1}}{m+1} + \frac{n-1}{m+1}\{\int_\theta(\sin\theta)^m(\cos\theta)^{n-2} - \int_\theta(\sin\theta)^m(\cos\theta)^n\};$$

$$\therefore\ \left(1 + \frac{n-1}{m+1}\right)u = \frac{m+n}{m+1}u = \frac{(\sin\theta)^{m+1}(\cos\theta)^{n-1}}{m+1}$$

$$+ \frac{n-1}{m+1}.\int_\theta(\sin\theta)^m(\cos\theta)^{n-2};$$

$$\therefore\ u = \frac{(\sin\theta)^{m+1}(\cos\theta)^{n-1}}{m+n} + \frac{n-1}{m+n}.\int_\theta(\sin\theta)^m(\cos\theta)^{n-2},$$

a formula by which the integral may be reduced to

$$\int_\theta(\sin\theta)^m,\ \text{or}\ \int_\theta(\sin\theta)^m\cos\theta.$$

73. Let $\frac{du}{d\theta} = \frac{\sin^m\theta}{\cos^n\theta}$,

$$u = \int_\theta \frac{\sin^{m-1}\theta\sin\theta}{(\cos\theta)^n} = \frac{(\sin\theta)^{m-1}}{(n-1).(\cos\theta)^{n-1}} - \frac{m-1}{n-1}\int_\theta \frac{(\sin\theta)^{m-2}}{(\cos\theta)^{n-2}},$$

a formula by which the integral is reducible to a known form.

74. Let $\frac{du}{d\theta} = \theta^n.\sin\theta$.

$$\int_\theta\theta^n\ \sin\theta = -\theta^n\cos\theta + n.\int\theta^{n-1}\cos\theta,$$

$$\int_\theta\theta^{n-1}\cos\theta = +\theta^{n-1}\sin\theta - (n-1)\int_\theta\theta^{n-2}\sin\theta,$$

$$\int_\theta\theta^{n-2}\sin\theta = -\theta^{n-2}\cos\theta + (n-2)\int_\theta\theta^{n-3}\cos\theta,$$

$$\&c. \quad = \quad \&c. \qquad \&c.$$

$$\int_\theta\theta^n\sin\theta = -\theta^n\cos\theta + n\theta^{n-1}\sin\theta + n(n-1)\theta^{n-2}\cos\theta$$

$$- n(n-1)(n-2)\theta^{n-3}\sin\theta - \&c.$$

COR. Similarly may $\int_\theta \theta^n \cos\theta$ be found and shewn to be

$$= \theta^n \sin\theta + n\theta^{n-1}\cos\theta - n(n-1)\theta^{n-2}\sin\theta$$
$$- n(n-1)(n-2)\theta^{n-3}\cos\theta + \&c.$$

75. Let $\dfrac{du}{d\theta} = \theta^{-n}\sin\theta = \dfrac{\sin\theta}{\theta^n}$;

$$\therefore \int_\theta \frac{\sin\theta}{\theta^n} = -\frac{\sin\theta}{(n-1)\,\theta^{n-1}} + \frac{1}{n-1}\int_\theta \frac{\cos\theta}{\theta^{n-1}},$$

$$\text{and } \int_\theta \frac{\cos\theta}{\theta^{n-1}} = -\frac{\cos\theta}{(n-2)\,\theta^{n-2}} - \frac{1}{n-2}\cdot\int_\theta \frac{\sin\theta}{\theta^{n-2}};$$

$$\therefore \int_\theta \frac{\sin\theta}{\theta^n} = -\frac{\sin\theta}{(n-1)\theta^{n-1}} - \frac{\cos\theta}{(n-1)(n-2)\theta^{n-2}} - \frac{1}{(n-1)(n-2)}\cdot\int\frac{\sin\theta}{\theta^{n-2}}$$

$$= -\frac{\sin\theta}{(n-1)\theta^{n-1}} - \frac{\cos\theta}{(n-1)(n-2)\theta^{n-2}} + \frac{\sin\theta}{(n-1)(n-2)(n-3)\theta^{n-3}} + \&c.$$

by which the integral may be reduced to $\displaystyle\int_\theta \frac{\sin\theta}{\theta}$, (if n be an integer) $\displaystyle = \int_\theta \left\{1 - \frac{\theta^2}{2.3} + \frac{\theta^4}{2.3.4.5} - \&c.\right\} = \theta - \frac{\theta^3}{2.3^2} + \frac{\theta^5}{2.3.4.5^2} - \&c.$

a similar method applies to $\displaystyle\int_\theta \frac{\cos\theta}{\theta^n}$.

76. Integrate $\sin m\theta \,.\, \cos n\theta$, $\sin m\theta \,.\, \sin n\theta$, and $\cos m\theta \,.\, \cos n\theta$;

$$\because \sin m\theta \,.\, \cos n\theta = \tfrac{1}{2}\,.\,\{\sin(m+n)\,\theta + \sin(m-n)\,\theta\};$$

$$\therefore \int_\theta(\sin m\theta.\cos n\theta) = -\tfrac{1}{2}\,.\left\{\frac{\cos(m+n)\,\theta}{m+n} + \frac{\cos(m-n)\,\theta}{m-n}\right\}.$$

Also since $\cos m\theta \,.\, \cos n\theta = \tfrac{1}{2}\,.\,\{\cos(m+n)\,\theta + \cos(m-n)\,\theta\}$,

and $\sin m\theta \,.\, \sin n\theta = \tfrac{1}{2}\,.\,\{\cos(m-n)\,\theta - \cos(m+n)\,\theta\}$;

$$\therefore \int_\theta(\cos m\theta \,.\, \cos n\theta) = \tfrac{1}{2}\,.\left\{\frac{\sin(m+n)\,\theta}{m+n} + \frac{\sin(m-n)\,\theta}{m-n}\right\},$$

$$\text{and } \int_\theta(\sin m\theta \sin n\theta) = -\tfrac{1}{2}\,.\left\{\frac{\sin(m+n)\,\theta}{m+n} - \frac{\sin(m-n)\,\theta}{m-n}\right\}.$$

COR. Similarly if $\dfrac{du}{d\theta} = \sin(a + m\theta) \,.\, \cos(b + n\theta)$,

put for $\sin(a+m\theta) \,.\, \cos(b+n\theta)$ its equivalent expression $\tfrac{1}{2}\,[\sin\{a + b + (m+n)\,\theta\} + \sin\{a - b + (m-n)\,\theta\}]$.

77. Integrate $(\tan\theta)^m$, and $(\tan\theta)^{-m}$.

$$(\tan\theta)^m = (\tan\theta)^{m-2}\{1+\tan^2\theta-1\}$$

$$= (\tan\theta)^{m-2}\frac{d.\tan\theta}{d\theta} - (\tan\theta)^{m-2};$$

$$\therefore \int_\theta(\tan\theta)^m = \frac{(\tan\theta)^{m-1}}{m-1} - \int_\theta(\tan\theta)^{m-2},$$

$$\int_\theta(\tan\theta)^{m-2} = \frac{(\tan\theta)^{m-3}}{m-3} - \int_\theta(\tan\theta)^{m-4},$$

&c. &c.

$$\therefore \int_\theta(\tan\theta)^m = \frac{(\tan\theta)^{m-1}}{m-1} - \frac{(\tan\theta)^{m-3}}{m-3} + \frac{(\tan\theta)^{m-5}}{m-5} - \&c.$$

by which u is reduced either to θ, or $\int_\theta \tan\theta = -\log\cos\theta$.

78. $$u = \int_\theta \frac{1}{(\tan\theta)^m} = \int_\theta \frac{1+\tan^2\theta-\tan^2\theta}{(\tan\theta)^m}$$

$$= \int_\theta \frac{\frac{d.(\tan\theta)}{d\theta}}{(\tan\theta)^m} - \int_\theta \frac{1}{(\tan\theta)^{m-2}} = -\frac{1}{(m-1)(\tan\theta)^{m-1}} - \int_\theta \frac{1}{(\tan\theta)^{m-2}}$$

and $$\int_\theta \frac{1}{(\tan\theta)^{m-2}} = -\frac{1}{(m-3)(\tan\theta)^{m-3}} - \int_\theta \frac{1}{(\tan\theta)^{m-4}};$$

$$\therefore \int_\theta \frac{1}{(\tan\theta)^m} = -\frac{1}{(m-1)(\tan\theta)^{m-1}}$$

$$+\frac{1}{(m-3)(\tan\theta)^{m-3}} - \frac{1}{(m-5)(\tan\theta)^{m-5}} + \&c.$$

and thus u may be reduced to θ, or $\int_\theta \frac{1}{\tan\theta} = \log(\sin\theta)$.

79. $\int_x e^{ax}\sin kx$.

$$\int_x e^{ax}\sin kx = \frac{e^{ax}\sin kx}{a} - \frac{k}{a}.\int_x e^{ax}\cos kx \ldots\ldots\ldots\ldots(1),$$

and $$\int_x e^{ax}\cos kx = \frac{e^{ax}\cos kx}{a} + \frac{k}{a}.\int_x e^{ax}\sin kx.$$

Multiplying by $\frac{k}{a}$, and transposing,

$$\frac{k^2}{a^2}.\int_x e^{ax}\sin kx = -\frac{ke^{ax}\cos kx}{a^2} + \frac{k}{a}.\int_x e^{ax}\cos kx \ldots\ldots\ldots(2).$$

Adding (1) and (2),

$$\left(1+\frac{k^2}{a^2}\right)\int_x e^{ax}\sin kx = \frac{(a\,.\sin kx - k\cos kx)e^{ax}}{a^2};$$

$$\therefore\ \int_x e^{ax}\sin kx = \frac{(a\,.\sin kx - k\cos kx)e^{ax}}{k^2+a^2}.$$

80. To integrate $\dfrac{du}{dx}=\dfrac{1}{a+b\,.\cos x}$

$$=\frac{1}{a\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)+b\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$$

$$=\frac{1}{(a+b)\,.\cos^2\frac{x}{2}+(a-b).\sin^2\frac{x}{2}}=\frac{\sec^2\frac{x}{2}}{a+b+(a-b)\,.\tan^2\frac{x}{2}}.$$

$$\text{Let}\quad z=\tan\frac{x}{2};$$

$$\therefore\ \frac{du}{dz}=\frac{2}{a+b+(a-b)z^2}.$$

(1) Let $a>b$; $\therefore\ u=\dfrac{2}{\sqrt{a^2-b^2}}\left(\tan^{-1}z\sqrt{\dfrac{a-b}{a+b}}\right)$

$$=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left\{\sqrt{\frac{a-b}{a+b}}\,.\tan\frac{x}{2}\right\}.$$

(2) Let $a<b$; $\therefore\ \dfrac{du}{dz}=\dfrac{2}{(b-a)}\cdot\dfrac{1}{\dfrac{b+a}{b-a}-z^2};$

$$\therefore\ u=\frac{1}{b-a}\cdot\sqrt{\frac{b-a}{b+a}}\,.\log\frac{\sqrt{\dfrac{b+a}{b-a}}+z}{\sqrt{\dfrac{b+a}{b-a}}-z}$$

$$=\frac{1}{\sqrt{b^2-a^2}}\,.\log\frac{\sqrt{b+a}+\sqrt{b-a}\,.\tan\frac{x}{2}}{\sqrt{b+a}-\sqrt{b-a}\,.\tan\frac{x}{2}}.$$

81. Similarly may $\displaystyle\int_x\frac{1}{a+b\sin x}$ be found.

Also $\int_x \frac{\sin x}{a+b\cos x} = \frac{1}{b}\int_x \frac{-\frac{d(b\cos x)}{dx}}{a+b\cos x} = -\frac{1}{b}\cdot\log(a+b\cos x)$,

and $\int_x \frac{\cos x}{a+b\cos x} = \int_x \frac{a+b\cos x - a}{b(a+b\cos x)}$

$$=\int_x\left\{\frac{1}{b}-\frac{a}{b}\cdot\frac{1}{a+b\cos x}\right\} = \frac{x}{b}-\frac{a}{b}\cdot\int_x\frac{1}{a+b\cos x}.$$

82. Integrate $\frac{du}{dx} = \frac{1}{a+b(\cos x)^2}$.

$$\int_x \frac{1}{a+b(\cos x)^2} = \int_x \frac{\sec^2 x}{a\sec^2 x + b}$$

$$=\int_x \frac{\sec^2 x}{a+b+a\tan^2 x} = \frac{1}{\sqrt{a^2+ab}}\cdot\tan^{-1}\left(\sqrt{\frac{a}{a+b}}\cdot\tan x\right).$$

83. If $\frac{du}{dx} = \frac{1}{a+b\tan x}$. Let $\tan x = z$; $\therefore \frac{dx}{dz} = \frac{1}{1+z^2}$;

$$\therefore \int_x \frac{1}{a+b\tan x} = \int_z \frac{1}{(1+z^2)(a+bz)}$$

$$=\frac{1}{a^2+b^2}\{b\cdot\log(a\cos x + b\sin x) + ax\}.$$

84. If $\frac{du}{dx} = \frac{a'+b'\cos x}{(a+b\cos x)^m}$.

Let $u = \frac{A\sin x}{(a+b\cos x)^{m-1}} + \int_x \frac{B+C\cos x}{(a+b\cos x)^{m-1}}$;

$\therefore$ differentiating and omitting the denominators,

$$a'+b'\cos x = A\cos x(a+b\cos x) + (m-1)Ab\sin^2 x$$
$$+(B+C\cos x)(a+b\cos x);$$

$$\therefore (m-1)Ab + Ba - a' + (Aa + Bb + Ca - b')\cos x$$
$$-\{(m-2)A - C\}\cos^2 x = 0,$$

$$\therefore A = \frac{ab'-ba'}{(m-1)(a^2-b^2)};\quad B = \frac{aa'-bb'}{a^2-b^2};\quad C = (m-2)A;$$

$$\therefore u = \frac{(ab'-ba')\sin x}{(m-1)(a^2-b^2)(a+b\cos x)^{m-1}}$$
$$+\int_x \frac{(m-1)(aa'-bb') + (m-2)(ab'-a'b)\cos x}{(m-1)(a^2-b^2)(a+b\cos x)^{m-1}}.$$

COR. 1. If $b'=0$ and $a'=1$, $\int_x \frac{1}{(a+b\cos x)^m}$

$$=\frac{1}{(m-1)(a^2-b^2)}\left\{\frac{-b\sin x}{(a+b\cos x)^{m-1}}\right.$$

$$\left.+\int_x \frac{(m-1)a-(m-2)b\cos x}{(a+b\cos x)^{m-1}}\right\}.$$

COR. 2. If $a'=0$ and $b'=1$; $\int_x \frac{\cos x}{(a+b\cos x)^m}$

$$=\frac{1}{(m-1)(a^2-b^2)}\left\{\frac{a\sin x}{(a+b\cos x)^{m-1}}-\int_x \frac{(m-1)b-(m-2)a\cos x}{(a+b\cos x)^{m-1}}\right\}.$$

85. Integrate $\int_x (a+b\cos x)^m$ by means of multiple arcs.

$$(a+b\cos x)^m = a^m\left(1+\frac{b}{a}\cos x\right)^m = a^m(1+n\cos x)^m;\quad n=\frac{b}{a};$$

$$\text{but } (1+n\cos x)^m = \{1+\frac{n}{2}(e^{x\sqrt{-1}}+e^{-x\sqrt{-1}})\}^m$$

$$=1+\frac{mn}{2}(e^{x\sqrt{-1}}+e^{-x\sqrt{-1}})+\frac{m(m-1)}{1.2}\frac{n^2}{4}(e^{x\sqrt{-1}}+e^{-x\sqrt{-1}})^2$$

$$+\frac{m(m-1).(m-2)}{1.2.3}\frac{n^3}{8}.(e^{x\sqrt{-1}}+e^{-x\sqrt{-1}})^3+\&c.$$

$$=1+mn\cos x+\frac{m(m-1)}{1.2}.\frac{n^2}{4}(2\cos 2x+2)$$

$$+\frac{m(m-1).(m-2)}{1.2.3}\frac{n^3}{8}(2\cos 3x+6\cos x)$$

$$+\frac{m(m-1).(m-2).(m-3)}{1.2.3.4}\frac{n^4}{16}(2\cos 4x+8\cos 2x+6)+\&c.$$

$$=A_0+A_1\cos x+A_2\cos 2x+A_3\cos 3x+A_4\cos 4x+\&c.$$

where $A_0=1+\frac{m(m-1)}{2.2}n^2+\frac{m(m-1).(m-2).(m-3).n^4}{2.2.4.4}+\&c.$

$$A_1=2n\left\{\frac{m}{2}+\frac{m(m-1).(m-2)n^2}{2.2.4}+\frac{m\ldots(m-4).n^4}{2.2.4.4.6}+\&c.\right\};$$

to find A_2, A_3, &c., take the logarithms and differentiate;

$$\therefore\ \frac{mn\sin x}{1+n\cos x}=\frac{A_1\sin x+2A_2\sin 2x+3A_3\sin 3x+\&c.}{A_0+A_1\cos x+A_2\cos 2x+\&c.}.$$

Then $\because$ $\sin x\cos \alpha x=\frac{1}{2}\{\sin(\alpha+1)x-\sin(\alpha-1)x\}$

$\cos x\sin\beta x=\frac{1}{2}\{\sin(\beta+1)x+\sin(\beta-1)x\}$;

therefore multiplying out and arranging the terms according to the sines of the multiple arc;

$$0=(A_1+\frac{2A_2}{2}n-mA_0n+\frac{m}{2}A_2n)\sin x$$

$$+(2A_2+\frac{1}{2}A_1n+\frac{3A_3}{2}n-\frac{m}{2}A_1n+\frac{m}{2}A_3n)\sin 2x$$

$$+(3A_3+\frac{2}{2}A_2n+\frac{4A_4}{2}n-\frac{m}{2}A_2n+\frac{m}{2}A_4n)\sin 3x$$

$$+\ \&c.;$$

$$\therefore\ A_2=\frac{2mnA_0-2A_1}{(m+2)n};\quad A_3=\frac{(m-1)A_1n-4A_2}{(m+3)n};$$

$$A_4=\frac{(m-2)A_2n-6A_3}{(m+4)n},$$

hence if A_0, A_1 are known, the other coefficients are also known.

86. When $m=-1$, or $u=\int_x \frac{1}{1+n\cos x}$.

$$A_0=1+\frac{1}{2}n^2+\frac{1.3}{2.4}n^4+\&c.=\frac{1}{\sqrt{1-n^2}};$$

$$A_1=2n(-\frac{1}{2}-\frac{1.3}{2.4}n^2-\&c.)=\frac{2}{n}.\left(1-\frac{1}{\sqrt{1-n^2}}\right)=\frac{2}{n}(1-A_0),$$

$$A_2=-\frac{2}{n}.(nA_0+A_1);\quad A_3=-\frac{1}{n}.(A_1n+2A_2);$$

$$A_4=-\frac{1}{n}(A_2n+2A_3);$$

similarly we may find the coefficients when $m=-\frac{1}{2}$, $m=-\frac{3}{2}$; the latter case is useful in Physical Astronomy.

87. Let $\frac{du}{dx}=\log(1+n\cos x)$.

$$\log(1+n\cos x)=n\cos x-\tfrac{1}{2}n^2\cos^2 x+\tfrac{1}{3}n^3\cos^3 x-\tfrac{1}{4}n^4\cos^4 x+\&c.$$

$$=-\frac{1}{2}\frac{n^2}{2}-\frac{1.3}{2.4}\frac{n^4}{4}-\frac{1.3.5}{2.4.6}\frac{n^6}{6}-\&c.$$

$$+(n+\frac{3}{4}\frac{n^3}{3}+\frac{3.5}{4.6}\frac{n^5}{5}+\&c.)\cos x+\&c.$$

$$=-A_0+A_1\cos x-A_2\cos 2x+A_3\cos 3x+\&c.\ (1);$$

where $A_0 = \frac{1}{2}\frac{n^2}{2} + \frac{1.3}{2.4}\frac{n^4}{4} + \frac{1.3.5}{2.4.6}\cdot\frac{n^6}{6} + \&c.;$

$$\therefore\ n\frac{dA_0}{dn} = \frac{1}{2}.n^2 + \frac{1.3}{2.4}.n^4 + \frac{1.3.5}{2.4.6}.n^6 + \&c. = \frac{1}{\sqrt{1-n^2}} - 1;$$

$$\therefore\ \frac{dA_0}{dn} = \frac{1}{n\sqrt{1-n^2}} - \frac{1}{n};\ \therefore\ A_0 = \log\frac{1-\sqrt{1-n^2}}{n} + \log\frac{1}{n} + C,$$

and $C = \log 2$, for $A_0 = 0$, when $n = 0$;

$$\therefore\ A_0 = \log\left(\frac{2-2\sqrt{1-n^2}}{n^2}\right),$$

$$\text{and } A_1 = n + \frac{1.3}{2.4}\cdot\frac{n^3}{3} + \frac{1.3.5}{4.6}\cdot\frac{n^5}{5} + \&c.;$$

$$\therefore\ \frac{dA_1}{dn} = \frac{2}{n^2}\left\{\frac{n^2}{2} + \frac{1.3}{4}\cdot\frac{n^4}{2} + \frac{1.3.5}{4.6}\cdot\frac{n^6}{2} + \&c.\right\}$$

$$= \frac{2}{n^2\sqrt{1-n^2}} - \frac{2}{n^2};$$

$$\therefore\ A_1 = 2\left(\frac{1-\sqrt{1-n^2}}{n}\right) + C = 2\left(\frac{1-\sqrt{1-n^2}}{n}\right);\ \because\ C = 0;$$

$$\therefore\ \text{if } \frac{1-\sqrt{1-n^2}}{n} = N;\ A_0 = \log\left(\frac{2N}{n}\right);\ A_1 = 2N;$$

and to find A_2, A_3, &c. differentiate (1);

$$\therefore\ \frac{n\sin x}{1+n\cos x} = A_1\sin x - 2A_2\sin 2x + 3A_3\sin 3x - 4A_4\sin 4x + \&c.$$

$$\therefore\ A_2 = \frac{A_1 - n}{n};\quad A_3 = \frac{4A_2 - A_1 n}{3n};\quad A_4 = \frac{6A_3 - 2A_2 n}{4n};$$

$$\therefore\ A_2 = \frac{2}{2}N^2,\ A_3 = \frac{2}{3}N^3,\ A_4 = \frac{2}{4}N^4,\ A_5 = \frac{2}{5}N^5,\ \&c.;$$

$$\therefore\ \textstyle\int_x \log(1+n\cos x) = \int_x\{\log\left(\frac{n}{2N}\right) + 2N\cos x - \frac{2}{2}N^2\cos 2x + \&c.\}$$

$$= x\log\frac{n}{2N} + 2N\sin x - \frac{2}{2.2}N^2.\sin 2x + \frac{2}{3.3}N^3.\sin 3x - \&c.$$

COR. If $n=1$; $\therefore$ $N=1$, and $\log(1+\cos x) = \log\left(2\cos^2\frac{x}{2}\right)$;

$$\therefore \ = \log 2 + 2 \log \cos \frac{x}{2}; \text{ hence putting } 2x \text{ for } x,$$

$$\log \cos x = \log \tfrac{1}{2} + \cos 2x - \tfrac{1}{2} \cos 4x + \tfrac{1}{3} \cos 6x - \&c.$$

And $n=-1$; $\therefore$ $N=-1$, and $\log(1-\cos x) = \log 2 - 2 \log \sin \frac{x}{2}$

$$= -\log \tfrac{1}{2} - 2 \log \sin \frac{x}{2}; \quad \therefore \text{ putting } 2x \text{ for } x,$$

$$\log \sin x = \log \tfrac{1}{2} - \cos 2x - \tfrac{1}{2} \cos 4x - \tfrac{1}{3} \cos 6x - \&c.;$$

$$\therefore \int_x \log \cos x = x \log \tfrac{1}{2} + \tfrac{1}{2} \sin 2x - \frac{1}{2.4} \sin 4x + \frac{1}{3.6} \sin 6x - \&c.$$

$$\int_x \log \sin x = x \log \tfrac{1}{2} - \tfrac{1}{2} \sin 2x - \frac{1}{2.4} \sin 4x - \frac{1}{3.6} \sin 6x - \&c.$$

and $\int_x \log \tan x = -\sin 2x - \frac{1}{9} \sin 6x - \&c.$

Examples.

(1) $\int_\theta (\sin\theta)^3 = -\tfrac{1}{3} \cdot (\sin\theta)^2 \cdot \cos\theta - \tfrac{2}{3} \cos\theta.$

(2) $\int_\theta (\cos\theta)^3 = \tfrac{1}{3} \cdot (\cos\theta)^2 \cdot \sin\theta + \tfrac{2}{3} \cdot \sin\theta.$

(3) $$\int_\theta (\sin\theta)^5 = -\cos\theta \left\{ \frac{(\sin\theta)^4}{5} + \frac{4(\sin\theta)^2}{15} + \frac{8}{15} \right\}.$$

(4) $$\int_\theta (\cos\theta)^6 = \sin\theta \left\{ \frac{(\cos\theta)^5}{6} + \frac{5(\cos\theta)^3}{24} + \frac{5\cos\theta}{16} \right\} + \frac{5\theta}{16}.$$

(5) $$\int_\theta (\sin\theta)^3 . (\cos\theta)^2 = \frac{\cos\theta}{5} \{ (\sin\theta)^4 - \tfrac{1}{3} (\sin\theta)^2 - \tfrac{2}{3} \}.$$

(6) $$\int_\theta (\sin\theta)^2 (\cos\theta)^4 = \frac{(\sin\theta . \cos\theta)^3}{6} + \frac{1}{8} \sin^3\theta \cos\theta - \frac{1}{16} \sin\theta \cos\theta + \frac{\theta}{16}.$$

(7) $$\int_\theta (\sin\theta)^6 (\cos\theta)^3 = \left\{ \frac{(\cos\theta)^2}{9} + \frac{2}{63} \right\} (\sin\theta)^7.$$

(8) $$\int_\theta \frac{1}{(\sin\theta)^3} = -\tfrac{1}{2} \cdot \frac{\cos\theta}{(\sin\theta)^2} + \tfrac{1}{2} \log\left(\tan\frac{\theta}{2}\right).$$

(9) $\int_\theta \frac{1}{(\sin\theta)^5} = -\cos\theta\left\{\frac{1}{4(\sin\theta)^4} + \frac{3}{8(\sin\theta)^2}\right\} + \frac{3}{8}\log\tan\frac{\theta}{2}.$

(10) $\int_\theta \frac{1}{(\cos\theta)^6} = \sin\theta\left\{\frac{1}{5(\cos\theta)^5} + \frac{4}{15(\cos\theta)^3} + \frac{8}{15.\cos\theta}\right\}.$

(11) $\int_\theta \frac{(\sin\theta)^3}{(\cos\theta)^4} = \frac{1}{(\cos\theta)^3}\{(\sin\theta)^2 - \frac{2}{3}\}.$

(12) $\int_\theta \frac{(\sin\theta)^5}{(\cos\theta)^2} = -\frac{1}{\cos\theta}\left\{\frac{(\sin\theta)^4}{3} + \frac{4(\sin\theta)^2}{3} - \frac{8}{3}\right\}.$

(13) $\int_\theta \frac{(\cos\theta)^4}{(\sin\theta)^3} = \left\{(\cos\theta)^2 - \frac{3\cos\theta}{2}\right\}\frac{1}{(\sin\theta)^2} - \frac{3}{2}.\log\tan\frac{\theta}{2}.$

(14) $\int_\theta \frac{1}{(\sin\theta)^2.(\cos\theta)^3} = \left\{\frac{1}{2(\cos\theta)^2} - \frac{3}{2}\right\}\frac{1}{\sin\theta}$

$$+\frac{3}{2}.\log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right).$$

(15) $\int_\theta \frac{1}{(\sin\theta)^4(\cos\theta)^2} = -\frac{1}{3\cos\theta(\sin\theta)^3} - \frac{8}{3}\cot 2\theta.$

(16) $\int_\theta (\tan\theta)^4 = \frac{1}{3}\tan^3\theta - \tan\theta + \theta.$

(17) $\int_\theta \frac{1}{(\tan\theta)^5} = \frac{-1}{4(\tan\theta)^4} + \frac{1}{2(\tan\theta)^2} + \log(\sin\theta).$

(18) $\int_\theta \theta^3.\cos\theta = \theta^3\sin\theta + 3\theta^2\cos\theta - 6\theta\sin\theta - 6\cos\theta.$

(19) $\int_x \frac{x^2}{\sqrt{1-x^2}}\sin^{-1}x = \frac{(\sin^{-1}x)^2}{4} - \frac{x\sqrt{1-x^2}}{2}\sin^{-1}x + \frac{x^2}{4}.$

(20) $\int_x \frac{x}{(1-x^2)^{\frac{3}{2}}}\sin^{-1}x = \frac{\sin^{-1}x}{\sqrt{1-x^2}} + \log\sqrt{\frac{1-x}{1+x}}.$

(21) $\int_x \frac{x^2}{1+x^2}\tan^{-1}x = x\tan^{-1}x - \frac{1}{2}(\tan^{-1}x)^2 - \log\sqrt{1+x^2}.$

(22) $\int_x x^2\tan^{-1}\sqrt{\frac{x}{a}} = \frac{x^3+a^3}{3}\tan^{-1}\sqrt{\frac{x}{a}}$

$$-\frac{\sqrt{ax}}{3}\left(\frac{x^2}{5} - \frac{ax}{3} + a^2\right).$$

(23) $\int_x e^{ax}.\cos kx = \frac{e^{ax}(a\cos kx + k\sin kx)}{k^2+a^2}.$

(24) $\int_x e^{-ax} \cdot \sin kx = -\dfrac{e^{-ax} \cdot (a \sin kx + k \cos kx)}{k^2 + a^2}$.

(25) $\int_x e^{ax} \cdot (\sin x)^2 = \dfrac{e^{ax} \cdot \sin x (a \sin x - 2 \cos x)}{a^2 + 4} + \dfrac{2 \cdot e^{ax}}{a(a^2 + 4)}$.

(26) $\displaystyle\int_\theta \frac{1}{a \cos^2 \theta + b \sin^2 \theta} = \frac{1}{\sqrt{ab}} \cdot \tan^{-1} \left(\sqrt{\frac{b}{a}} \tan \theta \right)$.

(27) $\displaystyle\int_\theta \frac{\cos \theta}{(1 - e^2 \cos^2 \theta)^{\frac{3}{2}}} = \frac{1}{1 - e^2} \cdot \frac{\sin \theta}{\sqrt{1 - e^2 \cos^2 \theta}}$.

(28) $\displaystyle\int_x \frac{1}{(a + b \cos x)^2} = \frac{1}{a^2 - b^2} \left\{ \frac{-b \sin x}{a + b \cos x} + a \int_x \frac{1}{a + b \cos x} \right\}$.

(29) $\int_\theta (\cos 2\theta)^{\frac{3}{2}} \cos \theta = \frac{1}{8} \sin \theta \, (3 + 2 \cos 2\theta) \sqrt{\cos 2\theta}$

$$+ \frac{3}{8\sqrt{2}} \sin^{-1} (\sqrt{2} \sin \theta).$$

(30) $\displaystyle\int_\theta \frac{\sin \theta}{\sqrt{\sin^2 \alpha - \sin^2 \theta}} = \log \cdot \left(\frac{\cos \alpha}{\cos \theta + \sqrt{\sin^2 \alpha - \sin^2 \theta}} \right)$

$$\left. \begin{array}{r} \text{from } \theta = \theta \\ \text{to } \theta = \alpha \end{array} \right\}.$$

(31) $\int_x e^{2A \cos x}$, (where $A = \log a$), between $x = 0$, and $x = \pi$

$$= \{1 + A^2 + \left(\frac{A^2}{2}\right)^2 + \left(\frac{A^3}{2 \cdot 3}\right)^2 + \&c.\} \, \pi.$$

(32) $\displaystyle\int_x e^{-ax} \cdot \sin kx \left\{ \begin{array}{c} \text{from } x = 0 \\ \text{to } x = \infty \end{array} \right\} = \frac{k}{k^2 + a^2}$.

CHAPTER VI.

Application of the Integral Calculus to determine the Areas and Lengths of Plane Curves, and the Volumes and Surfaces of Solids of Revolution.

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88. We have seen in the Differential Calculus, that if $y=f(x)$ be the equation to a curve, and A the area of a portion ANP, that $\frac{dA}{dx}=y=f(x)$. Hence, when the equation to a curve is given, its area may be found by finding the value of $\int_x f(x)$, and this integral may in general be found by means of the rules given in the preceding chapters. If the equation to the curve be between polar coordinates, then

$$\frac{dA}{d\theta}=\frac{r^2}{2}; \quad \therefore\ A=\int_\theta \frac{r^2}{2}.$$

It is sometimes convenient to substitute z for $\phi(x)$; but then, since $y=f(z)$,

$$\frac{dA}{dz}=\frac{dA}{dx}\cdot\frac{dx}{dz}=y\frac{dx}{dz};$$

$$\therefore\ A=\int_z y\cdot\frac{dx}{dz}=\int_z f(z)\cdot\frac{dx}{dz}.$$

89. Again, if s represents the length of a curve, of which the equation is $y=f(x)$,

$$\text{since } \frac{ds}{dx}=\sqrt{1+\frac{dy^2}{dx^2}}; \quad \therefore\ s=\int_x\sqrt{1+\frac{dy^2}{dx^2}},$$

where $\frac{dy}{dx}$ may be found from $y=f(x)$.

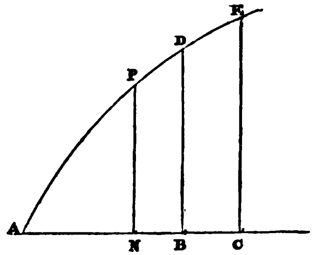
90. Also, if V and S respectively represent the volume and surface of a solid of revolution, since

$$\frac{dV}{dx}=\pi y^2, \text{ and } \frac{dS}{dx}=2\pi y\sqrt{1+\frac{dy^2}{dx^2}};$$

$$\therefore\ V=\pi\int_x y^2, \text{ and } S=2\pi\cdot\int_x y\sqrt{1+\frac{dy^2}{dx^2}}.$$

91. A constant must be added to each of these integrals, the determination of which depends upon the nature of the particular problem.

As an illustration, let the area ABD be required, the nature of the curve ANP being known by the equation $y=f(x)$, where $AN=x$, and $NP=y$.



Let

$AB=a$, and $ANP=A$;

$$\therefore \frac{dA}{dx}=y=f(x);$$

$$\therefore A=ANP=\int_x f(x)=\phi(x)+C\ldots\ldots(1).$$

To find C, we observe that if $x=0$ the area $=0$; if therefore at the same time $\phi(x)=0$; $\therefore C=0$,

$$\text{and } ANP=\phi(x), \text{ and } ABD=\phi(a);$$

the same result would be obtained had we successively put $x=0$ and $x=a$ in equation (1), and subtracted the former result from the latter. This process is called integrating between the limits of $x=0$ and $x=a$, and is commonly represented by the symbol $\int_0^a f(x)$; the first limit being placed below, the second above the sign of integration.

To take a second instance, let the area $DBCE$ be required where $AC=b$; putting a for x in equation (1),

$$\text{area } ABD=\phi(a)+C,$$

$$\text{and area } ACE=\phi(b)+C;$$

$$\therefore \text{area } BDEC=\phi(b)-\phi(a).$$

Hence, if the value of an integral $u=\phi(x)$ be required between two values a and b of x, omit the constant, and having put a and b successively for x in $\phi(x)$, subtract $\phi(a)$ from $\phi(b)$. This is called integrating between the limits or values a and b of x, and the integral so found is called a *definite integral*, and is expressed by $\int_a^b f(x)$.

We have already found definite integrals in the preceding pages, and if we use the symbol mentioned above,

$$\int_0^1 \frac{x^{2n}}{\sqrt{1-x^2}}=\frac{1.3.5\ldots(2n-3).(2n-1)}{2.4.6\ldots 2n-2\ .\ 2n}\cdot\frac{\pi}{2}. \quad \text{Art. (46)};$$

$$\int_0^{2a} \frac{x^m}{\sqrt{2ax-x^2}}=\frac{1.3.5\ldots(2m-3)(2m-1)}{2.4.6\ldots(m-1)\ .\ m}.\pi a^m. \quad \text{Art. (50)};$$

$$\int_{-\infty}^{\infty} e^{-t^2} = \sqrt{\pi}.$$ Art. (63), and some others.

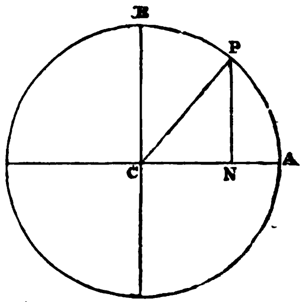
When C is not determined, the value, $\phi(x) + C$, is caldel the *general* integral.

Areas of Curves.

92. To find the areas of curves, or to integrate

$$\frac{dA}{dx} = y, \quad \text{or} \quad \frac{dA}{d\theta} = \frac{r^2}{2}.$$

Ex. 1. To find the area of the circle.



$$\left.\begin{array}{l} CN = x \\ NP = y \\ CA = a \end{array}\right\}; \quad \therefore\ y = \sqrt{a^2 - x^2};$$

$$\therefore\ A = \int_x y = \int_x \sqrt{a^2 - x^2};$$

$$\therefore\ \text{area } CBPN = \int_x \sqrt{a^2 - x^2}.$$

But $$\int_x \sqrt{a^2 - x^2} = \int_x \frac{\sqrt{a^2 - x^2}}{a^2 - x^2}$$

$$= \frac{a^2}{2} \sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + C.$$

$$C = 0, \text{ since area} = 0; \text{ if } x = 0;$$

$\sin^{-1}\frac{x}{a}$ can only be approximated to, by means of an infinite series, but if $x = a$, it $= \frac{\pi}{2}$, and,

$$\text{quadrant } ACB = \int_0^a \sqrt{a^2 - x^2} = \frac{a^2}{2} \cdot \frac{\pi}{2} = \frac{\pi a^2}{4};$$

$$\therefore\ \text{area of the circle} = \pi a^2.$$

Cor. 1. If $AN = x$, $y = \sqrt{2ax - x^2}$,

$$\therefore\ ANP = \int_0^x \sqrt{2ax - x^2}.$$

and when $x = a$, ANP becomes a quadrant;

$$\therefore\ \int_0^a \sqrt{2ax - x^2} = \frac{\pi a^2}{4}.$$

The two definite integrals $\int_0^a \sqrt{a^2 - x^2}$, and $\int_0^a \sqrt{2ax - x^2}$, should be carefully remembered.

Also $\int_x \sqrt{a^2 - x^2}$ being $= CBPN$, where CN is sometimes called the cosine to radius CA; $\therefore\ \int_x \sqrt{a^2 - x^2}$ is called a circular area of which cosine $= x$ and radius $= a$; and $\int_x \sqrt{2ax - x^2}$ in which $AN = x$, is called a circular area, of which ver. sine $= x$ and radius $= a$.

If $AN =$ the diameter, the area ANP is a semicircle;

$$\therefore\ \int_0^{2a} \sqrt{2ax - x^2} = \frac{\pi a^2}{2}.$$

Also $\because\ \int_0^{-a} \sqrt{a^2 - x^2} =$ area of the second quadrant;

$$\therefore\ \int_{-a}^{a} \sqrt{a^2 - x^2} = \frac{\pi a^2}{2}.$$

COR. 2. To find the area of the sector ACP.

Let $A =$ area ACP; $\theta = \angle\, ACP$;

$$\therefore\ \frac{dA}{d\theta} = \tfrac{1}{2} r^2 = \tfrac{1}{2} a^2;$$

$$\therefore\ A = \frac{a^2\theta}{2} = a \times \frac{a\theta}{2} = \frac{\text{radius} \times \text{arc}}{2}.$$

Ex. 2. To find the area of an ellipse.

The centre the origin; $CN = x$; $NP = y$,

$$\therefore\ y = \frac{b}{a} \sqrt{a^2 - x^2};$$

$$\therefore\ A = \int_x y = \frac{b}{a} \cdot \int_x \sqrt{a^2 - x^2};$$

$$\text{elliptic quadrant} = \frac{b}{a} \int^a \sqrt{a^2 - x^2} = \frac{b}{a} \cdot \frac{\pi a^2}{4} = \frac{\pi ab}{4};$$

$$\therefore\ \text{area of ellipse} = \pi ab.$$

Had the vertex been the origin, and $AN = x$,

$$\text{elliptic quadrant} = \frac{b}{a} \int_0^a \sqrt{2ax - x^2} = \frac{b}{a} \frac{\pi a^2}{4} = \frac{\pi ab}{4}.$$

Ex. 3. To find the area of the common parabola.

$$y^2 = 4mx;\quad \therefore\ y = 2\sqrt{mx}.$$

$$A = \int_x y = 2 \int_x \sqrt{mx} = 2\sqrt{m} \cdot \tfrac{2}{3} x^{\frac{3}{2}} + C;$$

$$\therefore \text{ area} = \int_0^x y = \frac{4\sqrt{m}}{3}\, x^{\frac{3}{2}} = \tfrac{2}{3}\, 2\sqrt{mx}\, .\, x = \tfrac{2}{3}\, yx$$

$$= \tfrac{2}{3} \text{ of circumscribing rectangle.}$$

Ex. 4. To find the area of the Witch.

$$y = \frac{2a}{x}\sqrt{2ax - x^2};$$

$$\therefore \text{ area} = \int_x y = 2a \int_x \frac{\sqrt{2ax - x^2}}{x} = 2a \int_x \frac{2a - x}{\sqrt{2ax - x^2}}$$

$$= 2a \left\{ \int_x \frac{a - x}{\sqrt{2ax - x^2}} + a \int_x \frac{1}{\sqrt{2ax - x^2}} \right\}$$

$$= 2a \left\{ \sqrt{2ax - x^2} + a \text{ ver-sin}^{-1} \frac{x}{a} \right\} + C.$$

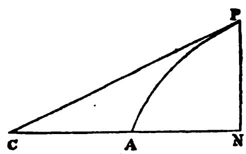
And area $= 0$, if $x = 0$; $\therefore\ C = 0$;

$$\therefore \text{ area} = 2a \left\{ \sqrt{2ax - x^2} + a \text{ ver-sin}^{-1} \frac{x}{a} \right\}.$$

Let $x = 2a$; $\therefore$ area $= 2a \times a\, .\, \pi = 2\pi a^2$.

Ex. 5. Find the area of the hyperbolic sector CAP.

Sector $CAP =$
$\triangle CNP -$ area ANP.



Let $\left.\begin{aligned} CN &= x \\ NP &= y \\ CA &= a \end{aligned}\right\}$;

$$\therefore\ y = \frac{b}{a}\sqrt{x^2 - a^2}.$$

$$ANP = \int_x y = \frac{b}{a} \int_x \sqrt{x^2 - a^2} = \frac{b}{a} \cdot \int_x \frac{x^2 - a^2}{\sqrt{x^2 - a^2}}$$

$$= \frac{b}{a} \cdot \left\{ \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \cdot \log\left(x + \sqrt{x^2 - a^2}\right) \right\} + C,$$

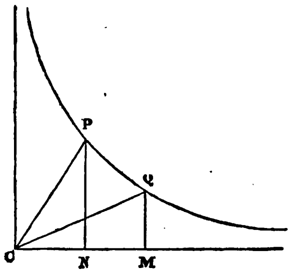
and $0 = -\dfrac{ba}{2} \cdot \log a + C$; $\because\ ANP = 0$; if $x = a$;

$$\therefore\ ANP = \frac{xy}{2} - \frac{ba}{2} \cdot \log\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right)$$

$$= \triangle CNP - \frac{ba}{2}.\log\left(\frac{x+\sqrt{x^2-a^2}}{a}\right);$$

$$\therefore \text{ sector } CAP = \frac{ba}{2}.\log\left(\frac{x}{a}+\frac{y}{b}\right).$$

Ex. 6. Find the area of the portion $PNMQ$, PQ being an arc of the rectangular hyperbola, axes the asymptotes.



Here $yx = \frac{a^2}{2}$.

Let $CN = \alpha$, and $CM = \beta$,

$$\int_x y = \frac{a^2}{2}\int_x \frac{1}{x} = \frac{a^2}{2}.\log x + C;$$

$$\therefore PQMN = \frac{a^2}{2}.(\log\beta - \log\alpha) = \frac{a^2}{2}.\log\left(\frac{\beta}{\alpha}\right).$$

Cor. Since $\frac{yx}{2} = \frac{a^2}{4}$; $\therefore \triangle CNP = \triangle CQM$;

$$\therefore \text{ sector } CPQ = \text{area } PNMQ.$$

Ex. 7. Find the area of the cycloid.

Origin from the vertex, $\frac{dy}{dx} = \frac{\sqrt{2ax-x^2}}{x}$,

$$\text{area} = \int_x y = yx - \int_x x\frac{dy}{dx} = yx - \int_x \sqrt{2ax-x^2},$$

also $\because$ if $x=0$, $y=0$; if $x=2a$, $y=\pi a$;

$$\therefore \int_0^{2a} y = \text{semicycloid} = 2\pi a^2 - \tfrac{1}{2}\pi a^2 = \frac{3\pi a^2}{2};$$

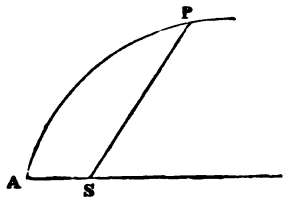
$\therefore$ cycloid $= 3\pi a^2 = 3$. area of generating circle.

Ex. 8. The area of the cissoid $= \frac{3\pi a^2}{2}$.

Ex. 9. Area of the conchoid $=$

$$ab\log.\left(\frac{x}{b+\sqrt{b^2-x^2}}\right) + \left(a+\frac{x}{2}\right)\sqrt{b^2-x^2} + \frac{b^2}{2}\sin^{-1}\left(\frac{x}{b}\right) + C.$$

Ex. 10. In the common parabola, to find the area ASP.



$$SA = a,$$
$$\angle ASP = \theta,$$
$$SP = r;$$

$$\therefore r = \frac{2a}{1+\cos\theta} = \frac{a}{\cos^2\frac{\theta}{2}};$$

$$\therefore A = \tfrac{1}{2}\int_\theta r^2 = a^2 . \int_\theta \frac{\frac{1}{2}}{\cos^4\frac{\theta}{2}} = a^2 . \int_\theta \frac{\frac{1}{2}}{\cos^2\frac{\theta}{2}}\left(1+\tan^2\frac{\theta}{2}\right)$$

$$= a^2\left\{\tan\frac{\theta}{2} + \frac{1}{3}\tan^3\frac{\theta}{2}\right\}.$$

Ex. 11. The area of the lemniscata $= a^2$.

Ex. 12. Find the area of the spiral where $r = a\theta^n$.

$$\text{Here } \frac{dA}{d\theta} = \tfrac{1}{2}r^2; \quad \theta = \left(\frac{r}{a}\right)^{\frac{1}{n}};$$

$$\therefore \frac{d\theta}{dr} = \frac{1}{na^{\frac{1}{n}}}r^{\frac{1}{n}-1}; \quad \therefore \frac{dA}{dr} = \frac{1}{2na^{\frac{1}{n}}}r^{1+\frac{1}{n}};$$

$$\therefore A = \frac{1}{2na^{\frac{1}{n}}} . \frac{n}{2n+1}r^{\frac{2n+1}{n}} + C,$$

and $C = 0$, if $A = 0$, when $r = 0$.

Cor. Let $n = 1$, or the spiral be that of Archimedes;

$$\therefore \text{area} = \frac{r^3}{6a} = \frac{\pi r^3}{3R},$$

if $R = r$ when $\theta = 2\pi$;

therefore area of spiral in first revolution $= \dfrac{\pi R^2}{3}$.

The area after two revolutions of the radius vector is when $\theta = 4\pi$, or when $r = 2R$. But before $r = 2R$, it will have made two revolutions, and therefore have twice generated the area from $r = 0$ to $r = R$.

Consequently we must subtract the area described in the first revolution from that in the second;

$$\therefore \text{ area} = \frac{\pi \cdot (2R)^3}{3R} - \frac{\pi R^2}{3} = \frac{7\pi R^2}{3}.$$

And the space between the arcs of the first and second areas

$$= \frac{7\pi R^2}{3} - \frac{\pi R^2}{3} = 2\pi R^2.$$

At the n^{th} revolution $r = nR$,

$\ldots\ldots.(n-1)^{\text{th}} \ldots\ldots\ldots\quad r = (n-1)R$;

$$\therefore \text{ area after } n \text{ revolutions} = \frac{\pi}{3} \cdot \frac{(nR)^3 - (n-1)^3 R^3}{R}$$

$$= \frac{\pi R^2}{3}\{n^3 - (n-1)^3\}.$$

$$\text{Area after } (n+1) \text{ revolutions} = \frac{\pi R^2}{3}\{(n+1)^3 - n^3\};$$

$\therefore$ space between the arcs after $n+1$ and n revolutions

$$= \frac{\pi R^2}{3} \cdot \{(n+1)^3 + (n-1)^3 - 2n^3\} = \frac{\pi R^2}{3} \cdot 6n = 2n\pi R^2$$

$= n$ times the space between the first and second.

Ex. 13. Find the area of the involute of the circle, where $r^2 - p^2 = a^2$.

$$\frac{dA}{d\theta} = \frac{1}{2} r^2 \frac{d\theta}{dr} = \frac{p}{r\sqrt{r^2 - p^2}};$$

$$\therefore \frac{dA}{dr} = \frac{pr}{2\sqrt{r^2 - p^2}} = \frac{r\sqrt{r^2 - a^2}}{2a};$$

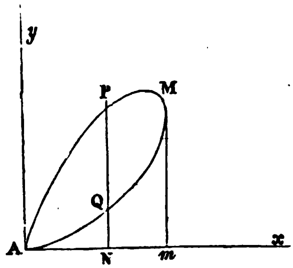
$$\therefore A = \frac{(r^2 - a^2)^{\frac{3}{2}}}{6a} + C; \text{ and } C = 0.$$

If $p = 2\pi a$; $A = \frac{4}{3}\pi^3 a^2$, or subtracting πa^2, the area of the involute exterior of the circle after one unwrapping of the string $= \pi a^2 \left(\frac{4\pi^2}{3} - 1\right)$.

Ex. 14. Find the area of the curve of which the equation is

$$y^3 - 3axy + x^3 = 0.$$

The curve has a nodus $APMQ$, Ay and Ax being tangents.



$$\text{Let } y = xz; \quad \therefore\ z = \frac{y}{x} = \tan PAN;$$

$$\therefore\ x^3z^3 - 3ax^2z + x^3 = 0;$$

$$\therefore\ x = \frac{3az}{1+z^3}, \text{ and } y = \frac{3az^2}{1+z^3};$$

and since x is $= 0$, for each of the branches APM and AQM, this will happen if $z = \infty$ or $= 0$.

$$\text{Now } \frac{dA}{dz} = \frac{dA}{dx} \cdot \frac{dx}{dz} = y \cdot \frac{dx}{dz},$$

$$\text{and } \frac{dx}{dz} = \frac{3a \cdot \{1 + z^3 - 3z^3\}}{(1+z^3)^2} = 3a \frac{1-2z^3}{(1+z^3)^2};$$

$$\therefore\ A = \int_z y \cdot \frac{dx}{dz} = 9a^2 \cdot \int_z \frac{z^2(1-2z^3)}{(1+z^3)^3}$$

$$= 9a^2 \int_z \left\{ \frac{z^2[1 - 2(z^3+1) + 2]}{(1+z^3)^3} \right\}$$

$$= 9a^2 \int_z \left\{ \frac{3z^2}{(1+z^3)^3} - \int_z \frac{2z^2}{(1+z^3)^2} \right\}$$

$$= 9a^2 \left\{ -\tfrac{1}{2} \cdot \frac{1}{(1+z^3)^2} + \frac{2}{3} \cdot \frac{1}{1+z^3} \right\} + C.$$

$$\text{Let } z = 0; \quad \therefore\ C = -\frac{3a^2}{2}, \text{ and let } z_1 = \frac{y}{x} \text{ at } M;$$

$$\therefore\ \text{area } AQMm = -\frac{3a^2}{2} + 9a^2 \left\{ -\tfrac{1}{2} \cdot \frac{1}{(1+z_1^3)^2} + \frac{2}{3} \frac{1}{1+z_1^3} \right\}.$$

Integrating $z = \infty$ and $z = z_1$ for the branch APM,

$$\text{area } APMm = 9a^2 \left\{ -\tfrac{1}{2}\frac{1}{(1+z_1^3)^2} + \frac{2}{3}\cdot\frac{1}{1+z_1^3} \right\};$$

$\therefore$ the nodus $APMQ =$ area $APMm -$ area $AQMm = \dfrac{3a^2}{2}$.

If the area of the nodus only be required, then,

$$\text{since } \tan\theta = \frac{y}{x} = z; \quad \therefore \frac{d\theta}{dz} = \cos^2\theta;$$

$$\therefore \frac{dA}{dz} = \tfrac{1}{2}r^2\frac{d\theta}{dz} = \tfrac{1}{2}r^2\cos^2\theta = \frac{x^2}{2};$$

$$\therefore A = \tfrac{1}{2}\textstyle\int_z x^2; \text{ and } x^2 = \dfrac{9a^2z^2}{(1+z^3)^2};$$

$$\therefore A = \frac{9a^2}{2}\cdot\int_z \frac{z^2}{(1+z^3)^2} = C - \frac{3a^2}{2}\cdot\frac{1}{1+z^3},$$

integrating from $z = 0$, to $z = \infty$: area $= \dfrac{3a^2}{2}$.

Ex. 15. To find the area of the evolute of an ellipse.

$$\text{Here } \left(\frac{x}{\alpha}\right)^{\frac{2}{3}} + \left(\frac{y}{\beta}\right)^{\frac{2}{3}} = 1.$$

We might put $y = xz$; but better thus:

$$\text{Let } \frac{x}{\alpha} = \cos^3\theta; \quad \therefore \frac{y}{\beta} = \sin^3\theta; \quad \frac{dx}{d\theta} = -3\alpha\sin\theta\,.\,\cos^2\theta;$$

$$\therefore \frac{dA}{d\theta} = \frac{dA}{dx}\cdot\frac{dx}{d\theta} = y\frac{dx}{d\theta} = -3\alpha\beta\sin^4\theta\cos^2\theta;$$

$\therefore A = 3\alpha\beta\,.\int_\theta \sin^4\theta\cos^2\theta$; the limits of which are $\theta = \dfrac{\pi}{2}$ and $\theta = 0$, since those of x are, 0 and α.

$$\text{Now } \int_\theta \sin^4\theta\cos^2\theta = \frac{\cos\theta\sin^5\theta}{5} + \frac{1}{5}\int_\theta \sin^6\theta;$$

$$\text{but } \frac{\cos\theta\,.\,\sin^5\theta}{5} = 0, \text{ both when } \theta = 0 \text{ and } \theta = \frac{\pi}{2};$$

$$\therefore \int_{\frac{\pi}{2}}^{0} \sin^4\theta\cos^2\theta = -\frac{1}{5}\int_{\frac{\pi}{2}}^{0} \sin^6\theta = -\frac{1}{5}\times\frac{5\,.\,3\,.\,1}{6\,.\,4\,.\,2}\cdot\frac{\pi}{2};$$

$$\therefore A = 3\alpha\beta\,.\,\frac{3\pi}{6\times4\times4} = \frac{3\alpha\beta}{8}\cdot\frac{\pi}{4};$$

$$\therefore 4A = \text{whole area} = \frac{3\pi\alpha\beta}{8} = \frac{3}{8}\pi\frac{(a^2-b^2)^2}{ab}.$$

Ex. 16. The same substitution applies to find the area of

$$\left(\frac{y}{\beta}\right)^{\frac{2}{2n+1}} + \left(\frac{x}{a}\right)^{\frac{2}{2n+1}} = 1.$$

For making $x = a\cos^{2n+1}\theta$; $\therefore\ y = \beta\sin^{2n+1}\theta$,

$$\frac{dx}{d\theta} = -(2n+1)\,a\cos^{2n}\theta\sin\theta;$$

$$\therefore\ A = -(2n+1)a\beta\int_{\frac{\pi}{2}}^{0}\sin^{2n+2}\theta\cos^{2n}\theta$$

$$= -(2n+1)\,a\beta\int_{\frac{\pi}{2}}^{0}\{\sin^{2n+2}\theta\,(1-\sin^2\theta)^n\};$$

whence expanding, $\because \displaystyle\int_{\frac{\pi}{2}}^{0}\sin^{2n}\theta = -\frac{(2n-1).(2n-3)\ldots3.1}{2n.(2n-2)\ldots4.2}\cdot\frac{\pi}{2}$;

$$\therefore\ \text{area} = 4A = (4n+2)\pi a\beta\left\{\frac{1.3.5\ldots(2n+1)}{2.4.6\ldots(2n+2)} - n\cdot\frac{1.3\ldots(2n+3)}{2.4\ldots(2n+4)}\right.$$

$$\left. + n.\frac{(n-1)}{2}\cdot\frac{1.3\ldots(2n+5)}{2.4\ldots(2n+6)} - n\frac{(n-1)(n-2)}{2.3}\cdot\frac{1.3\ldots(2n+7)}{2.4\ldots(2n+8)} + \&c.\right\};$$

If $n = 1$, $\left(\frac{y}{\beta}\right)^{\frac{2}{3}} + \left(\frac{x}{a}\right)^{\frac{2}{3}} = 1$, and area $= 6\pi a\beta\left\{\frac{3}{8} - \frac{3.5}{8.6}\right\}$

$= \frac{3}{8}\pi a\beta$, the result obtained in the preceding example.

The lengths of Curves.

93. To find the lengths of curves, or to integrate

$$\frac{ds}{dx} = \sqrt{1 + \frac{dy^2}{dx^2}},\ \text{when } y = f(x).$$

Ex. 17. Find the length of an arc of the parabola.

$$s = \int_x\sqrt{1+\frac{dy^2}{dx^2}} = \int_x\sqrt{1+\frac{m}{x}} = \int_x\frac{\sqrt{x+m}}{\sqrt{x}}$$

$$= \int_x\frac{x+m}{\sqrt{x^2+mx}} = \int_x\frac{x+\frac{m}{2}}{\sqrt{x^2+mx}} + \int_x\frac{\frac{m}{2}}{\sqrt{x^2+mx}}$$

$$= \sqrt{x^2+mx} + \frac{m}{2}\log\left(x + \frac{m}{2} + \sqrt{x^2+mx}\right) + C.$$

And $s=0$, if $x=0$; $\therefore\ 0=\frac{m}{2}\log\left(\frac{m}{2}\right)+C$;

$$\therefore\ s=\sqrt{x^2+mx}+\frac{m}{2}\log\left(\frac{2x+m+2\sqrt{x^2+mx}}{m}\right).$$

Ex. 18. Find when curves included under the general equation $y=ax^{\frac{m}{n}}$ are rectifiable.

$$\frac{dy}{dx}=\frac{m}{n}ax^{\frac{m-n}{n}};\quad \therefore\ s=\int_x\sqrt{1+\frac{m^2a^2}{n^2}.x^{\frac{2m-2n}{n}}};$$

which is integrable.

(1) When $\frac{n}{2m-2n}$ is an integer $=r$,

$$\text{or } \frac{m}{n}-1=\frac{1}{2r},\ \text{ or } \frac{m}{n}=\frac{1}{2r}+1=\frac{2r+1}{2r}.$$

(2) When $\frac{n}{2m-2n}+\frac{1}{2}=$ an integer $=q$,

$$\text{or } \frac{m}{n}-1=\frac{1}{2q-1},\ \text{ or } \frac{m}{n}=\frac{2q}{2q-1}.$$

Let $r=1, 2, 3,$ &c. $q=1, 2, 3,$ &c.;

$$\therefore\ \frac{m}{n}=\frac{3}{2},\ \frac{5}{4},\ \frac{7}{6},\ \&c. \text{ and } \frac{m}{n}=\frac{2}{1},\ \frac{4}{3},\ \frac{6}{5},\ \&c.$$

Ex. 19. Let $\frac{m}{n}=\frac{3}{2}$; (the semi-cubical parabola)

$$\therefore\ y=ax^{\frac{3}{2}};\quad \frac{dy}{dx}=\frac{3a}{2}x^{\frac{1}{2}}=\frac{\sqrt{x}}{\sqrt{c}},\ \text{ if } \sqrt{c}=\frac{2}{3a};$$

$$\therefore\ s=\int_x\sqrt{1+\frac{x}{c}}=\frac{1}{\sqrt{c}}.\int_x\sqrt{x+c}=\frac{1}{\sqrt{c}}\frac{2}{3}(x+c)^{\frac{3}{2}}+C.$$

$$\text{If } s=0,\ x=0;\quad \therefore\ C=-\frac{1}{\sqrt{c}}\frac{2}{3}c^{\frac{3}{2}};$$

$$\therefore\ s=\frac{1}{\sqrt{c}}.\frac{2}{3}.\{(x+c)^{\frac{3}{2}}-c^{\frac{3}{2}}\}.$$

Ex. 20. Find the length of the cycloid.

$$\frac{dy}{dx}=\sqrt{\frac{2a-x}{x}};\quad \therefore\ 1+\frac{dy^2}{dx^2}=1+\frac{2a-x}{x}=\frac{2a}{x};$$

$$\therefore\ s=\int_x\sqrt{\frac{2a}{x}}=\sqrt{2a}.\int_x\frac{1}{\sqrt{x}}=2\sqrt{2ax}+C,$$

and $C=0$, since $s=0$, when $x=0$;

therefore $s = 2\sqrt{2ax}$ = twice the chord of the arc of the generating circle, corresponding to the arc of the cycloid.

Hence the cycloid is rectifiable.

If $x = 2a$, $s = 2\sqrt{4a^2} = 4a$, or the length of the semi-cycloid = twice the diameter of the circle.

Ex. 21. Find the length of the arc of an ellipse.

$$y = \frac{b}{a}\sqrt{a^2 - x^2}; \quad \frac{dy}{dx} = -\frac{b}{a}\cdot\frac{x}{\sqrt{a^2 - x^2}};$$

$$\therefore\ 1 + \frac{dy^2}{dx^2} = 1 + \frac{b^2x^2}{a^2(a^2 - x^2)} = \frac{a^4 - (a^2 - b^2)x^2}{a^2(a^2 - x^2)} = \frac{a^2 - e^2x^2}{a^2 - x^2};$$

$$\therefore\ s = \int_x \frac{\sqrt{a^2 - e^2x^2}}{\sqrt{a^2 - x^2}} = a\int_z \frac{\sqrt{1 - e^2z^2}}{\sqrt{1 - z^2}}, \text{ (if } x = za)$$

$$= a\int_z \frac{1}{\sqrt{1 - z^2}}\cdot\{1 - \tfrac{1}{2}e^2z^2 - \frac{1.1}{2.4}e^4z^4 - \frac{1.1.3}{2.4.6}\cdot e^6z^6 - \&c.\},$$

the integration of which depends on $\int_z \frac{z^{2n}}{\sqrt{1 - z^2}}$.

If the quadrant be required, we must integrate from $x = 0$ to $x = a$, or from $z = 0$ to $z = 1$, but then

$$\int_0^1 \frac{z^{2n}}{\sqrt{1 - z^2}} = \frac{\pi}{2}\cdot\frac{1.3.5......(2n-1)}{2.4.6......2n};$$

$$\therefore\ \int_0^1 \frac{z^2}{\sqrt{1 - z^2}} = \frac{\pi}{2}\cdot\frac{1}{2}; \quad \int_0^1 \frac{z^4}{\sqrt{1 - z^2}} = \frac{\pi}{2}\cdot\frac{1.3}{2.4};$$

$$\int_0^1 \frac{z^6}{\sqrt{1 - z^2}} = \frac{\pi}{2}\cdot\frac{1.3.5}{2.4.6}, \ \&c.; \quad \int_0^1 \frac{1}{\sqrt{1 - z^2}} = \frac{\pi}{2};$$

therefore elliptic quadrant

$$= \frac{\pi a}{2}\cdot\{1 - \frac{1}{2^2}e^2 - \frac{1.3}{2^2.4^2}e^4 - \frac{1.3^2.5}{2^2.4^2.6^2}e^6 - \&c.\},$$

a series rapidly convergent when e is a small fraction.

(3) To find the same by circular functions.

Let $x = a\cos\theta$; $\therefore\ y = b\sin\theta$, $\because\ x$ is $< a$;

$$\therefore\ \frac{ds}{d\theta} = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$$

$$= \sqrt{a^2 - (a^2 - b^2)\cos^2\theta} = a\sqrt{1 - e^2\cos^2\theta}$$

$$= a\{1 - \tfrac{1}{2}e^2\cos^2\theta - \frac{1.1}{2.4}e^4\cos^4\theta - \frac{1.1.3}{2.4.6}e^6\cos^6\theta - \&c.\}.$$

Now $\int_\theta \cos^{2n}\theta = \sin\theta\,.\cos^{2n-1}\theta + (2n-1)\,.\int_\theta \cos^{2n-2}\theta\,.\sin^2\theta$

$$=\frac{\sin\theta\cos^{2n-1}\theta}{2n}+\frac{2n-1}{2n}\,.\int_\theta \cos^{2n-2}\theta,$$

and $\sin\theta\cos^{2n-1}\theta = 0$, when $\theta = 0$, and $\theta = \frac{\pi}{2}$;

$\therefore$ calling $\int_0^{\frac{\pi}{2}} \cos^{2n}\theta = P_{2n}$,

$$P_{2n}=\frac{2n-1}{2n}\,.P_{2n-2}=\frac{(2n-1)(2n-3)}{2n(2n-2)}P_{2n-4}$$

$$=\frac{(2n-1).(2n-3)\ldots 3\,.\,1}{2n\,.(2n-2)\ldots 4\,.\,2}\,.\,\frac{\pi}{2};\quad \because\ P_0=\frac{\pi}{2}.$$

$$\therefore\ P_2=\tfrac{1}{2}\,.\,\frac{\pi}{2};\quad P_4=\frac{1\,.\,3}{2\,.\,4}\,.\,\frac{\pi}{2},\quad P_6=\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\,.\,\frac{\pi}{2}\ \&c.;$$

$$\therefore\ s=\frac{\pi a}{2}\,.\left\{1-\frac{1}{2^2}e^2-\frac{1\,.\,3}{2^2.4^2}e^4-\frac{1\,.\,3^2.\,5}{2^2.4^2.6^2}\,.e^6-\frac{1\,.\,3^2.\,5^2.\,7}{2^2.4^2.6^2.8^2}e^8-\&c.\right\}.$$

Ex. 22. Find the length of a hyperbolic arc.

$$y=\frac{b}{a}\sqrt{x^2-a^2},\quad \frac{dy}{dx}=\frac{b}{a}\,.\,\frac{x}{\sqrt{x^2-a^2}};$$

$$\therefore\ \frac{ds}{dx}=\sqrt{1+\frac{dy^2}{dx^2}}=\sqrt{\frac{(b^2+a^2)x^2-a^4}{a^2(x^2-a^2)}}=\sqrt{\frac{e^2x^2-a^2}{x^2-a^2}};$$

$$\therefore\ s=\int_x\sqrt{\frac{e^2x^2-a^2}{x^2-a^2}}=a\,.\int_z\sqrt{\frac{e^2z^2-1}{z^2-1}},\ (\text{if } x=az),$$

$$=ae\int_z z\frac{\sqrt{1-\frac{1}{e^2z^2}}}{\sqrt{z^2-1}}$$

$$=a\int_z\frac{ez}{\sqrt{z^2-1}}\,.\left\{1-\tfrac{1}{2}\frac{1}{(ez)^2}-\frac{1.1}{2.4}\,.\,\frac{1}{(ez)^4}-\frac{1.1.3}{2.4.6}\,.\,\frac{1}{(ez)^6}-\&c.\right\},$$

the limits being $x=a$; $x=\infty$, or $z=1$; $z=\infty$;

now every term, except the first, depends upon

$$\int_z\frac{1}{z^m\sqrt{z^2-1}},\ m \text{ odd},$$

and $\int_z \frac{1}{z^m\sqrt{z^2-1}} = \frac{1}{m-1}\cdot\frac{\sqrt{z^2-1}}{z^{m-1}} + \frac{m-2}{m-1}\cdot\int_z \frac{1}{z^{m-2}\sqrt{z^2-1}}$,

and $\frac{\sqrt{z^2-1}}{z^{m-1}}$ vanishes both when $z=1$, and $z=\infty$;

$$\therefore \int_1^\infty \frac{1}{z^m\sqrt{z^2-1}} = \frac{m-2}{m-1}\int_1^\infty \frac{1}{z^{m-2}\sqrt{z^2-1}}.$$

But $\int_1^\infty \frac{1}{z\sqrt{z^2-1}} = \sec^{-1}z = \frac{\pi}{2}$;

$$\therefore \int_1^\infty \frac{1}{z^3\sqrt{z^2-1}} = \tfrac{1}{2}\int_1^\infty \frac{1}{z\sqrt{z^2-1}} = \tfrac{1}{2}\cdot\frac{\pi}{2};$$

$$\int_1^\infty \frac{1}{z^5\sqrt{z^2-1}} = \tfrac{3}{4}\cdot\int_1^\infty \frac{1}{z^3\sqrt{z^2-1}} = \frac{1\,.\,3}{2\,.\,4}\cdot\frac{\pi}{2};$$

$$\therefore s = ae\,.\int_z \frac{z}{\sqrt{z^2-1}} - \frac{\pi a}{2}\,.\,\{\tfrac{1}{2}\,.\,\frac{1}{e} + \frac{1\,.\,1}{2^2\,.\,4}\cdot\frac{1}{e^3}$$

$$+ \frac{1\,.\,1\,.\,3^2}{2^2.\,4^2.\,6}\cdot\frac{1}{e^5} + \frac{1\,.\,1\,.\,3^2.\,5^2}{2^2.\,4^2.\,6^2.\,8}\cdot\frac{1}{e^7} + \&c.\}.$$

Now the equation to the asymptote is $y = \frac{bx}{a}$;

$\therefore$ length of asymptote $= \sqrt{x^2 + \frac{b^2x^2}{a^2}} = x\sqrt{\frac{a^2+b^2}{a^2}} = ex = aez.$

But $ae\int_z \frac{z}{\sqrt{z^2-1}} = ae\sqrt{z^2-1} = aez$ from $z=1$ to $z=\infty$.

If therefore l be the length of the asymptote,

$$l - s = \frac{\pi a}{2}\left\{\frac{1}{2}\cdot\frac{1}{e} + \frac{1\,.\,1}{2^2.\,4}\cdot\frac{1}{e^3} + \frac{1\,.\,1\,.\,3^2}{2^2.\,4^2.\,6}\cdot\frac{1}{e^5}\right.$$

$$\left. + \frac{1\,.\,1\,.\,3^2.\,5^2}{2^2.\,4^2.\,6^2.\,8}\cdot\frac{1}{e^7} + \&c.\right\}.$$

Ex. 23. Find the length of an arc of the logarithmic curve.

$$s = \frac{\sqrt{1+A^2y^2}}{A} + \frac{1}{A}\log\frac{Ay}{1+\sqrt{1+A^2y^2}} + C.$$

Ex. 24. Find the length of an arc of the Lemniscata.

$$r^2 = a^2 \cos 2\theta, \quad \text{and } s = \int_r \sqrt{1 + r^2 \frac{d\theta^2}{dr^2}};$$

$$\therefore s = \int_r \frac{a^2}{\sqrt{a^4 - r^4}} = a \cdot \int_z \frac{1}{\sqrt{1 - z^4}}, \text{ if } r = az$$

$$= a \int_z \cdot \left(\frac{1}{\sqrt{1 - z^2}} \cdot \frac{1}{\sqrt{1 + z^2}} \right)$$

$$= a \int_z \cdot \frac{1}{\sqrt{1 - z^2}} \cdot \left\{ 1 - \tfrac{1}{2} z^2 + \frac{1.3}{2.4} z^4 - \frac{1.3.5}{2.4.6} z^6 + \&c. \right\}.$$

Integrate from $\theta = 45^\circ$ to $\theta = 0$; or from $z = 0$ to $z = 1$;

$$\therefore \int_0^1 \frac{1}{\sqrt{1 - z^2}} = \frac{\pi}{2}, \quad \int_0^1 \frac{z^2}{\sqrt{1 - z^2}} = \tfrac{1}{2} \cdot \frac{\pi}{2};$$

$$\int_0^1 \frac{z^4}{\sqrt{1 - z^2}} = \frac{1.3}{2.4} \cdot \frac{\pi}{2}, \text{ and } \int_0^1 \frac{z^6}{\sqrt{1 - z^2}} = \frac{1.3.5}{2.4.6} \cdot \frac{\pi}{2};$$

$$\therefore s = \frac{\pi a}{2} \cdot \left\{ 1 - \frac{1}{2^2} + \frac{1^2 . 3^2}{2^2 . 4^2} - \frac{1^2 . 3^2 . 5^2}{2^2 . 4^2 . 6^2} + \frac{1^2 . 3^2 . 5^2 . 7^2}{2^2 . 4^2 . 6^2 . 8^2} - \&c. \right\}.$$

The whole length of the lemniscata $= 4s$.

Ex. 25. To find the length of the involute of a circle.

$$\text{Here } r^2 - p^2 = a^2; \quad \frac{ds}{dr} = \frac{r}{\sqrt{r^2 - p^2}} = \frac{r}{a};$$

$$\therefore s = \frac{r^2}{2a} + C = \frac{r^2 - a^2}{2a} = \frac{p^2}{2a} = \frac{a}{2} \cdot \theta^2.$$

If $\theta = 2\pi$, or the string be unwound once, $s = 2\pi^2 a$.

If $\theta = 2n\pi$, or the string be unwound n times length $= \frac{a}{2}(2n\pi)^2$.

Ex. 26. If the radius of the circle be unity, and θ be a circular arc, θ_1 the length of its involute, θ_2 the length of the involute of θ_1, &c., prove that

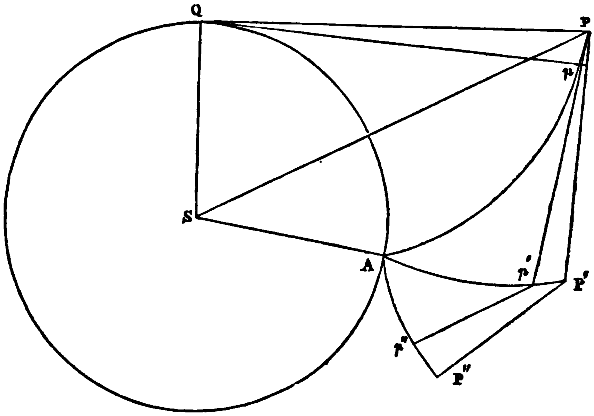
$$\theta + \theta_1 + \theta_2 + \&c. = e^\theta - 1.$$

$AQ = \theta, \qquad QP = \rho,$

$AP = \theta_1, \qquad PP_1 = \rho_1,$

$AP_1 = \theta_2, \qquad P_1P_2 = \rho_2,$

$AP_2 = \theta_3.$



Then $\angle PQp = \angle P_1Pp_1$; $\therefore \dfrac{Pp}{QP} = \dfrac{P_1p_1}{PP_1}$;

$$\therefore \frac{d\theta_1}{QP} \text{ or } \frac{d\theta_1}{\theta} = \frac{d\theta_2}{\theta_1}; \text{ but } \theta_1 = \tfrac{1}{2}\theta^2;$$

$$\therefore d\theta_2 = \frac{\theta_1}{\theta} d\theta_1 = \tfrac{1}{2}\theta^2 d\theta; \quad \therefore \theta_2 = \frac{1}{2.3}\theta^3.$$

$$\text{And } d\theta_3 = \frac{\theta_2}{\theta_1} d\theta_2 = \theta_2 . \frac{d\theta_1}{\theta} = \frac{1}{2.3} . \theta^3 . d\theta;$$

$$\therefore \theta_3 = \frac{1}{2.3.4}\theta^4;$$

$$\therefore \theta + \theta_1 + \theta_2 + \&c. = \theta + \tfrac{1}{2}\theta^2 + \frac{1}{2.3}\theta^3 + \&c. = e^\theta - 1.$$

The Volumes and Surfaces of Solids of Revolution.

94. To find the volumes and surfaces of solids, or to integrate the functions

$$\frac{dV}{dx} = \pi y^2, \text{ and } \frac{dS}{dx} = 2\pi y\sqrt{1 + \frac{dy^2}{dx^2}}.$$

Ex. 27. To find the volume of a cone with a circular base.

Let a = altitude, b = radius of base.

Then if the vertex be the origin and the altitude the axis of x, $y=\frac{b}{a}x$;

$$\therefore\ V=\pi\int_x y^2=\frac{\pi b^2}{a^2}\int_x x^2=\frac{\pi b^2}{a^2}\cdot\frac{x^3}{3}+C.$$

And $V=0$ if $x=0$; $\therefore\ C=0$; $\therefore\ V=\frac{\pi b^2}{a^2}\cdot\frac{x^3}{3}$.

Let $x=a$; $\therefore$ whole cone $=\frac{\pi b^2 a}{3}=\frac{1}{3}$ of a cylinder of the same altitude and on the same base.

Ex. 28. Find the volume of the paraboloid.

$y^2=4mx$ is the equation to the generating curve;

$$\therefore\ V=\pi\int_x y^2=\pi\int_x 4mx=2\pi m\,.\,x^2+C,\quad\text{and } C=0;$$

$$\therefore\ V=2\pi mx^2=\frac{\pi 4mx\,.\,x}{2}=\frac{\pi y^2 x}{2}.$$

But $\pi y^2 x=$ volume of a cylinder, base $=\pi y^2$ and altitude $=x$;

$\therefore$ paraboloid $=\frac{1}{2}$ circumscribing cylinder.

Ex. 29. Find the volume of a sphere.

Here $y^2=2ax-x^2$;

$$\therefore\ V=\pi\int_x.\,(2ax-x^2)=\pi\left(ax^2-\frac{x^3}{3}\right)+C,$$

and $V=0$ if $x=0$; $\therefore\ C=0$; $\therefore\ V=\pi x^2\left\{a-\frac{x}{3}\right\}$.

Let $x=2a$; $\therefore$ sphere $=4\pi a^2\left(a-\frac{2}{3}a\right)=\frac{4}{3}\pi a^3$.

Since circumscribing cylinder $=2a\,.\,\pi a^2=2\pi a^3$;

$\therefore$ sphere $=\frac{2}{3}$ of circumscribing cylinder.

Ex. 30. Find the volume of the prolate spheroid (formed by the revolution of an ellipse round its major axis).

$$y^2=\frac{b^2}{a^2}(a^2-x^2)\ldots\ldots(1);$$

$$\therefore\ V=\pi\int_x\frac{b^2}{a^2}(a^2-x^2)=\pi\frac{b^2}{a^2}\left(a^2x-\frac{x^3}{3}\right)+C$$

$$=\frac{4}{3}\pi b^2 a;\ \text{from } x=-a,\ \text{to } x=+a.$$

If the solid content of the oblate spheroid, which is formed by revolution round the minor axis be required,

take the minor axis for the axis of x, and the major for that of y.

Then in equation (1) put y for x and x for y, we have

$$x^2 = \frac{b^2}{a^2}(a^2 - y^2); \quad \therefore\ y^2 = \frac{a^2}{b^2}(b^2 - x^2);$$

$$\therefore\ \text{solid} = \pi \,.\int_x \frac{a^2}{b^2}(b^2 - x^2) = \frac{\pi a^2}{b^2}\left(b^2 x - \frac{x^3}{3}\right)$$

$$= \frac{4}{3}\pi a^2 b; \quad \text{from } x = -b, \quad \text{to } x = +b;$$

$\therefore$ prolate spheroid : oblate spheroid :: b : a.

COR. Sphere on major axis : prolate spheroid :: $a^2 : b^2$,
sphere on minor axis : oblate spheroid :: $b^2 : a^2$.

Ex. 31. Find the solid generated by the conchoid round the asymptote.

$$\text{Here } xy = (a + x)\sqrt{b^2 - x^2};$$

and since the curve revolves round the axis of y;

$$\therefore\ dV = \pi x^2 dy = -\pi \frac{ab^2 + x^3}{\sqrt{b^2 - x^2}}\,.\,dx;$$

$$\therefore\ V = -\pi \int_x \left\{\frac{ab^2}{\sqrt{b^2 - x^2}} + \frac{x^3}{\sqrt{b^2 - x^2}}\right\}$$

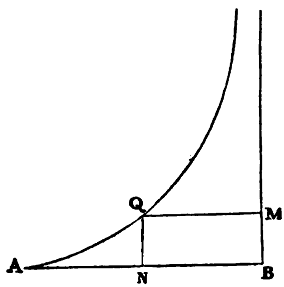
$$= C - \pi\,.\left\{ab^2 \sin^{-1}\frac{x}{b} - y^2\sqrt{b^2 - x^2} - \tfrac{2}{3}(b^2 - x^2)^{\frac{3}{2}}\right\},$$

and $x = b$; $V = 0$; $\therefore\ 0 = C - \pi\left\{ab^2\frac{\pi}{2}\right\}$; $\therefore\ C = \dfrac{\pi^2 ab^2}{2}$;

$$\therefore\ V = \frac{\pi^2 ab^2}{2} - \pi\left\{ab^2 \sin^{-1}\frac{x}{b} - \frac{\sqrt{b^2 - x^2}}{3}(x^2 + 2b^2)\right\}.$$

Let $x = 0$; $\therefore$ whole volume $= \pi b^2\left\{\dfrac{\pi a}{2} + \dfrac{2b}{3}\right\}$.

Ex. 32. Find the volume generated by the revolution of the cissoid round its asymptote.



$AB = 2a,\ BM = x,\ MQ = y.$

Now $NQ^2 = \dfrac{AN^3}{BN}$,

or $x^2 = \dfrac{(2a - y)^3}{y}$;

$$\therefore\ V = \pi \int_x y^2 = \pi y^2 x - 2\pi \int_y yx.$$

But $x^2y^2 = y(2a-y)^3$; $\therefore\ xy = \sqrt{y}\,.(2a-y)^{\frac{3}{2}}$;

$$\therefore\ \int_y xy = \int_y (2a-y)\sqrt{2ay-y^2}$$

$$= \int_y (a-y)\sqrt{2ay-y^2} + a\int_y \sqrt{2ay-y^2}$$

$$= \frac{(2ay-y^2)^{\frac{3}{2}}}{3} + a\int_y \sqrt{2ay-y^2};$$

$$\therefore\ V = \pi\left\{(2ay-y^2)^{\frac{3}{2}} - \tfrac{2}{3}(2ay-y^2)^{\frac{3}{2}} - 2a\int_y \sqrt{2ay-y^2}\right\};$$

$$\therefore\ \text{whole solid} = \int_{2a}^{0} y^2 = \pi\,.\,2a\,.\,\frac{\pi a^2}{2} = \pi^2 a^3.$$

Ex. 33. Find the solid generated by the revolution of the semi-cycloid round its base.

Make the base the axis of x; $\therefore\ \dfrac{dy}{dx} = \dfrac{\sqrt{2ay-y^2}}{y}$;

$$\therefore\ V = \int_y \pi y^2\,.\,\frac{dx}{dy} = \pi\,.\int_y \frac{y^3}{\sqrt{2ay-y^2}}\,.$$

But $\displaystyle\int \frac{y^3}{\sqrt{2ay-y^2}} = -\frac{y^2\sqrt{2ay-y^2}}{3} - \frac{5}{2\,.\,3}\,ay\sqrt{2ay-y^2}$

$$-\frac{3\,.\,5}{2\,.\,3}\,a^2\sqrt{2ay-y^2} + \frac{3\,.\,5}{2\,.\,3}\,a^3\,\text{ver-sin}^{-1}\frac{y}{a};$$

$$\therefore\ V = \pi\int_0^{2a} \frac{y^3}{\sqrt{2ay-y^2}} = \frac{5\pi^2 a^3}{2}.$$

Ex. 34. Find the solid generated by the revolution of the cycloid round its axis.

$$V = \pi\int_x y^2 = \pi\int_\theta y^2\frac{dx}{d\theta};$$

and $y = a(\theta + \sin\theta)$; $x = a(1-\cos\theta)$;

$$\therefore\ V = \pi a^3\int_\theta (\theta + \sin\theta)^2 \sin\theta$$

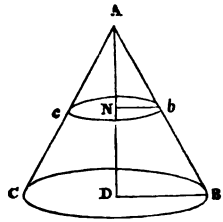
$$= \pi a^3\int_\theta \{\theta^2\sin\theta + 2\theta\sin^2\theta + \sin^3\theta\},$$

whence integrating from $\theta = 0$ to $\theta = \pi$;

$$\therefore\ V = \pi a^3\left\{\frac{3\pi^2}{2} - \frac{8}{3}\right\}.$$

Ex. 35. To find the volume of a conical figure, the base of which is bounded by a given curve.

From A draw AD perpendicular to the base, and $= a$. In AD take $AN = x$, N being a point in a section bc, parallel and similar to the base BC.



Let $A =$ area of the base,

$S =$ area of section bc;

$$\therefore \frac{S}{A} = \frac{bN^2}{BD^2} = \frac{AN^2}{AD^2} = \frac{x^2}{a^2};$$

$$\therefore S = A\frac{x^2}{a^2}, \text{ and } \frac{dV}{dx} = S = A \cdot \frac{x^2}{a^2};$$

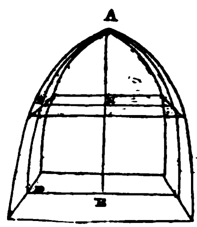
$$\therefore V = \frac{A}{a^2}\int_x x^2 = \frac{Ax^3}{3a^2} + C, \text{ and } C = 0;$$

$$\therefore ABC = \frac{Aa^3}{3a^2} = \frac{A \cdot a}{3} = \text{base} \times \tfrac{1}{3} \text{ of the altitude.}$$

Cor. This proposition is manifestly true for a pyramid of any base.

Ex. 36. To find the volume of a Groin; a solid of which in this instance, the sections parallel to the base are squares, and those perpendicular, bounded by a given curve.

Let the given curve AD be a quadrant



$AN = x$, $NP = y$; $AB = BD = a$;

therefore generating area $= (2y)^2 = 4y^2$;

$$\therefore \frac{dV}{dx} = 4y^2 = 4(2ax - x^2);$$

$$\therefore V = 4\left(ax^2 - \frac{x^3}{3}\right) = \tfrac{8}{3}a^3, \text{ if } x = a.$$

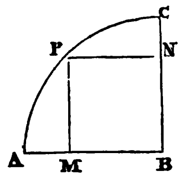
Again, $\because$ generating surface = perimeter of square $= 8y$;

$$\therefore \frac{dS}{dx} = 8y\frac{ds}{dx} = 8\sqrt{a^2 - x^2} \cdot \frac{a}{\sqrt{a^2 - x^2}} = 8a;$$

$$\therefore S = 8ax = 8a^2.$$

And similarly may the volume and surface be found, whatever be the curve APD. Also, if the base be any other figure, of which the area is a function of y, as a circle, a parabola, a triangle, &c. and APB be a curve of which the equation is $y = f(x)$, the surface and volume may be found.

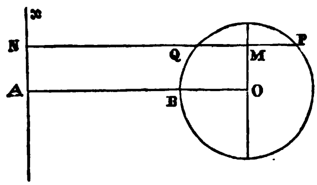
Ex. 37. Find the solid generated by a parabolic area round its ordinate.



$$AM = x, \qquad AB = a,$$
$$MP = y, \qquad BC = b;$$
$$\frac{PN^2}{AB^2} = \frac{CN}{CB}; \quad \frac{(a-x)^2}{a^2} = \frac{b-y}{b};$$
$$\therefore \; y = \frac{b}{a^2}(2ax - x^2);$$
$$\therefore \; V = \frac{8}{15}\pi a^2 b.$$

The double of this is a parabolic spindle.

Ex. 38. Find the volume and surface of the solid generated by the circle BQP round an axis ANx, in its own plane.



Let $AO = b$, $OB = a$,
$MQ = y$, $OM = x$.

Then surface generated by QP
$$= \pi\,(NP^2 - NQ^2) = \pi\{(b+y)^2 - (b-y)^2\} = 4\pi by;$$
$$\therefore \; \frac{dV}{dx} = 4\pi by; \quad \therefore \; V = 4\pi b \textstyle\int_x y = 4\pi b \dfrac{\pi a^2}{2} = 2\pi^2 a^2 b,$$
$$S = 2\pi . \textstyle\int_x (NP + NQ) . \dfrac{ds}{dx} = 4\pi b . \displaystyle\int_x \frac{ds}{dx} = 4\pi b . \pi a = 4\pi^2 ba.$$

Ex. 39. The surface of a sphere.

$$y = \sqrt{2ax - x^2}, \text{ and } \frac{dy}{dx} = \frac{a-x}{\sqrt{2ax - x^2}},$$
$$1 + \frac{dy^2}{dx^2} = 1 + \frac{(a-x)^2}{2ax - x^2} = \frac{a^2}{2ax - x^2} = \frac{a^2}{y^2},$$
$$S = 2\pi \textstyle\int_x y \sqrt{1 + \dfrac{dy^2}{dx^2}} = 2\pi \textstyle\int_x y . \dfrac{a}{y} = 2\pi \textstyle\int_x a = 2\pi a x + C,$$

$S = 0$, if $x = 0$; $\therefore$ $C = 0$; $\therefore$ surface of a segment $= 2\pi a x$;

$\therefore$ surface of sphere $= 2\pi a . 2a = 4\pi a^2$.

Ex. 40. Convex surface of a paraboloid.

$$\therefore \; S = \textstyle\int_x 2\pi y \sqrt{1 + \dfrac{dy^2}{dx^2}} = 4\pi \sqrt{m} . \textstyle\int_x \sqrt{x} \sqrt{\dfrac{x+m}{x}}$$
$$= 4\pi \sqrt{m} \textstyle\int_x \sqrt{x+m} = 4\pi \sqrt{m} \tfrac{2}{3} (x+m)^{\frac{3}{2}} + C,$$

and if $x=0$, $S=0$; $\therefore C=-\frac{8}{3}\pi\sqrt{m}.m^{\frac{3}{2}}$;

$$\therefore \text{ surface} = \frac{8\pi\sqrt{m}}{3}.\{(x+m)^{\frac{3}{2}}-m^{\frac{3}{2}}\}.$$

Ex. 41. Surface generated by a semi-cycloid round its base.

Here $\frac{dy}{dx}=\frac{\sqrt{2ay-y^2}}{y}$; $\therefore \sqrt{1+\frac{dy^2}{dx^2}}=\sqrt{\frac{2a}{y}}$;

$$\therefore \frac{dS}{dy}=\frac{dS}{dx}.\frac{dx}{dy}=2\pi\sqrt{2ay}.\frac{y}{\sqrt{2ay-y^2}};$$

$$\therefore S=2\pi\sqrt{2a}.\int_y\frac{y}{\sqrt{2a-y}}$$

$$=2\pi\sqrt{2a}\{-2y\sqrt{2a-y}-\frac{4}{3}(2a-y)^{\frac{3}{2}}\};$$

$$\therefore \text{ surface by semi-cycloid} = 2\pi.\frac{4}{3}(2a)^2=\frac{32}{3}\pi a^2.$$

Ex. 42. Find the same when round the axis.

The vertex the origin, $\frac{dy}{dx}=\sqrt{\frac{2a-x}{x}}$.

$$\text{Surface} = 2\pi\int_x y\frac{ds}{dx}=2\pi\left\{ys-\int_x s\frac{dy}{dx}\right\};\quad s=2\sqrt{2ax}$$

$$=4\pi\{y\sqrt{2ax}-\sqrt{2a}\int_x\sqrt{2a-x}\}$$

$$=4\pi\sqrt{2a}\{y\sqrt{x}+\frac{2}{3}(2a-x)^{\frac{3}{2}}\},$$

from $x=0$ to $x=2a$, or $y=0$ to $y=\pi a$,

$$S=4\pi\sqrt{2a}\{\pi a\sqrt{2a}-\frac{2}{3}.(2a)^{\frac{3}{2}}\}$$

$$=8\pi a\{\pi a-\frac{4}{3}a\}=8\pi a^2.\left\{\pi-\frac{4}{3}\right\}.$$

Ex. 43. To find the surface of the prolate spheroid.

$$y=\frac{b}{a}\sqrt{a^2-x^2}, \text{ and } \sqrt{1+\frac{dy^2}{dx^2}}=\sqrt{\frac{a^2-e^2x^2}{a^2-x^2}},$$

$$\frac{ds}{dx}=\frac{2\pi b}{a}\sqrt{a^2-e^2x^2}=2\pi b.\sqrt{1-\frac{e^2x^2}{a^2}};$$

$$\therefore S = \frac{\pi ba}{e} \cdot \left\{\sin^{-1}\left(\frac{ex}{a}\right) + \frac{ex}{a}\sqrt{1 - \frac{e^2x^2}{a^2}}\right\};$$

$$\therefore \left\{\begin{matrix}\text{from } x = -a \\ \text{to } x = +a\end{matrix}\right\} S = \frac{2\pi ba}{e} \cdot \{\sin^{-1} e + e\sqrt{1 - e^2}\}$$

$$= 2\pi a^2 \left\{\sqrt{1 - e^2} \cdot \frac{\sin^{-1} e}{e} + 1 - e^2\right\}.$$

Let $e = 0$, or spheroid become a sphere; $\therefore \dfrac{\sin^{-1} e}{e} = 1$,

and surface $= 2\pi a^2 \{1 + 1\} = 4\pi a^2$.

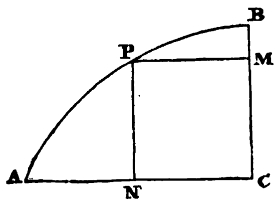
Ex. 44. To find the surface of an oblate spheroid.

$BM = x_1$ $CN = x$,

$MP = y_1$ $NP = y$,

$$\frac{dS}{dx_1} = 2\pi y_1 \cdot \frac{ds}{dx_1};$$

or $\dfrac{dS}{dy} = 2\pi x \cdot \dfrac{ds}{dy}$;



$$\therefore \frac{dS}{dx} = 2\pi x \frac{ds}{dx} = 2\pi x \sqrt{\frac{a^2 - e^2x^2}{a^2 - x^2}}.$$

Make $\sqrt{a^2 - x^2} = z$; and $c = \dfrac{b}{e}$;

$$\therefore S = -2\pi e \int_z \sqrt{c^2 + z^2}$$

$$= C - \pi e \{z\sqrt{c^2 + z^2} + c^2 \log(z + \sqrt{z^2 + c^2})\}.$$

From $x = 0$ to $x = a$; or from $z = a$ to $z = 0$,

$$S = \pi e \cdot \left\{a\sqrt{c^2 + a^2} + c^2 \log \frac{a + \sqrt{a^2 + c^2}}{c}\right\};$$

$$\therefore \text{surface} = 2S = 2\pi e \cdot \left\{a\sqrt{c^2 + a^2} + c^2 \log \frac{a + \sqrt{a^2 + c^2}}{c}\right\}$$

$$= 2\pi a^2 \cdot \left\{1 + \frac{(1 - e^2)}{2e} \log\left(\frac{1 + e}{1 - e}\right)\right\};$$

which $= 4\pi a^2$, when $e = 0$, or if the spheroid become a sphere.

(45) How much of the Earth's surface may be seen by a person elevated the $\frac{1}{n}$th part of the Earth's radius above it.

Ans. $\frac{1}{2n+2}$ th part.

(46) Find the length of the curve, where $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

$s=6a$.

(47) If h = height of a parabolic frustum, a and b the radii of the ends, shew that

$$\text{Frustum}=\frac{\pi h}{2}\cdot(a^2+b^2).$$

(48) Find the area of the catenary, $y=\frac{a}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})$.

$$\text{Area}=a\sqrt{y^2-a^2}.$$

(49) The area of $(x^2+y^2)^2=a^2x^2-b^2y^2$ is

$$=ab+(a^2-b^2)\tan^{-1}\left(\frac{a}{b}\right).$$

(50) The area of a parabolic segment, cut off by any chord $=\frac{2}{3}$ of circumscribing parallelogram.

(51) Content of sphere : content of greatest cone inscribed on it :: 27 : 8.

(52) Find the content of the least paraboloid about a given sphere.

(53) The area of the nodus in the curve defined by $y^4=a(x^3-2x^2y)$ is $\frac{a^2\cdot 2^6}{3\cdot5\cdot7}$.

(54) Find the area of $x^4y^4-a^4y^4=a^8$.

$$\text{Area}=\frac{a^2}{2}\left\{\log\sqrt{\frac{xy+a^2}{xy-a^2}}-\tan^{-1}\left(\frac{a^2}{xy}\right)\right\}.$$

(55) In a parabola, the area included between the curve, its evolute, and its radius of curvature.

$$=4\sqrt{\frac{x}{a}}\left\{a^2+\frac{2}{3}ax+\frac{1}{5}x^2\right\}.$$

(56) If the subtangent of the logarithmic curve = that of the spiral, $\theta = \frac{a}{r}$: the arc included by two radii of the spiral = arc included by two respectively equal ordinates of the curve.

(57) Find the length of the spiral of Archimedes.

$$s = \frac{r}{2a}\sqrt{r^2 + a^2} + \frac{a}{2} \cdot \log\left(\frac{r + \sqrt{r^2 + a^2}}{2a}\right).$$

(58) The length of the epicycloid after one revolution of the generating circle $= 8\frac{b}{a}(a+b)$, and the area between the epicycloid and the circle $= \pi b^2\left(3 + \frac{2b}{a}\right)$.

(59) The volume generated by the revolution of the Witch, round its asymptote $= 4\pi^2 a^3$.

(60) The area of the curve in which

$$r = \frac{(a^2 - b^2)\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}};\ \text{is} = \frac{\pi}{2}(a-b)^2.$$

(61) If equidistant ordinates be drawn in the hyperbola between the asymptotes, the contents of the solids generated by the included areas round the asymptote, will be as the fractions $\frac{1}{1.2}, \frac{1}{2.3}, \frac{1}{3.4}$, &c.

(62) In AP the chord of a semicircle take $AQ = PN$, then area of curve traced by $Q = \frac{\pi a^2}{8}$.

(63) If $r = a\sec\frac{\theta}{2}$, the area included by the curve, the asymptotes and tangent at vertex $= 4a^2$.

(64) If A = area of a logarithmic spiral from $r = 0$, to r: A_1 that of the locus of intersections of perpendiculars from origin on tangents, with tangents, A_2 of the curve similarly described, $A + A_1 + A_2 + \&c. = A \cdot \left(\frac{1+k^2}{k^2}\right)$; $k = \log_e a$.

CHAPTER VII.

Differential Equations.

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95. In the integrations which have been performed in the preceding Chapters, the differential coefficient has either been a given function of one of the variables, or else has been expressed in such terms of the two, that by a very evident process it has been reduced to a function of one only. We now proceed to integrate differentials, when the differential coefficients and the variables x and y are mingled together.

96. Differential equations are divided into classes, dependent upon the *order* and *degree* of the differential coefficient.

Thus an equation involving

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \text{ \&c. } \frac{d^ny}{dx^n},$$

is called a differential equation of the n^{th} order and of the first degree, while one containing

$$\frac{dy}{dx}, \left(\frac{dy}{dx}\right)^2, \left(\frac{dy}{dx}\right)^3, \text{ \&c. } \left(\frac{dy}{dx}\right)^n$$

is said to be of the first order, and of the n^{th} degree: and finally, an equation in which are to be found the n^{th} powers of the differential coefficients and the m^{th} differential coefficient, is named an equation of the m^{th} order and the n^{th} degree.

We shall begin with that class in which the first power of the first differential coefficient is alone found.

Differential Equations of the first Order and the first Degree.

97. These are included under the formula

$$M + N \cdot \frac{dy}{dx} = 0,$$

where M and N may be any functions of x and y; we shall however in the first place treat of homogeneous equations.

98. Let $M + N\dfrac{dy}{dx} = 0$, be a homogeneous equation, or one in which the sum of the indices of y and x together, is the same in every term.

$$\text{Make } y = xz;\ \therefore \frac{dy}{dx} = z + x\frac{dz}{dx}.$$

Divide by N and the equation becomes,

$$\frac{M}{N} + \frac{dy}{dx} = 0,\ \text{ or } \frac{M}{N} + z + x\frac{dz}{dx} = 0.$$

But $\dfrac{M}{N}$ is of no dimensions or is a function of $\dfrac{y}{x}$ or z.

$$\text{Let } \therefore \frac{M}{N} = f(z);\ \therefore x\frac{dz}{dx} = -\{z + f(z)\};$$

$$\therefore \frac{dx}{xdz} = -\frac{1}{z + f(z)};\ \therefore \log\left(\frac{x}{c}\right) = -\int_z \frac{1}{z + f(z)},$$

which may be integrated by the ordinary rules.

We put $x = yz$, or $y = xz$, as may be most convenient, for the solution is more easily effected, when we substitute for that differential coefficient which involves the fewest terms.

Ex. 1. Let $x + y = (x - y)\dfrac{dy}{dx}$.

$$\text{Here make } y = xz;\ \therefore \frac{dy}{dx} = z + x\frac{dz}{dx};$$

$$\therefore z + x\frac{dz}{dx} = \frac{x + y}{x - y} = \frac{1 + z}{1 - z};\ \therefore x\frac{dz}{dx} = \frac{1 + z^2}{1 - z};$$

$$\therefore \frac{dx}{xdz} = \frac{1 - z}{1 + z^2} = \frac{1}{1 + z^2} - \frac{z}{1 + z^2};$$

$$\therefore \log\left(\frac{x}{c}\right) = \tan^{-1} z - \log\sqrt{1 + z^2};$$

$$\therefore \log\left(\frac{x}{c}\sqrt{1 + z^2}\right),\ \text{ or } \log\frac{\sqrt{x^2 + y^2}}{c} = \tan^{-1}\frac{y}{x}.$$

Ex. 2. Find the curve in which the subtangent is equal to the sum of the abscissa and ordinate.

$$\text{Here } y\frac{dx}{dy} = x + y;\ \text{ and let } x = yz;$$

$$\therefore \frac{dx}{dy} = z + y\frac{dz}{dy} = \frac{x + y}{y} = z + 1;$$

$$\therefore \frac{dy}{ydz} = 1; \quad \therefore \log\left(\frac{y}{c}\right) = z = \frac{x}{y}.$$

Ex. 3. Find the curve in which the subnormal $= y - x$.

$$\therefore \log\left(\frac{\sqrt{y^2 - yx + x^2}}{c}\right) = \frac{1}{\sqrt{3}} \cot^{-1}\left(\frac{2y - x}{x\sqrt{3}}\right).$$

Ex. 4. Find the curve in which the distance from the origin to a point in the curve equals the subtangent.

$$\text{Here, } \sqrt{y^2 + x^2} = y\frac{dx}{dy}; \quad \text{let } x = yz;$$

$$\therefore \log\left\{\frac{y^3}{c^2(x + \sqrt{x^2 + y^2})}\right\} = \frac{x}{y^2}(x + \sqrt{x^2 + y^2}).$$

Ex. 5. $(\sqrt{x} - \sqrt{y}) = \sqrt{y} \cdot \frac{dx}{dy}$.

Ex. 6. $x\frac{dy}{dx} - y = \sqrt{x^2 + y^2}$; then $x^2 = c^2 + 2cy$.

Ex. 7. $x\frac{dy}{dx} - y = y \log\left(\frac{y}{x}\right)$; $\therefore y^c = x^c e^x$.

Ex. 8. $\int_x y = \frac{y^3}{x}$; $\therefore (x^2 - 2y^2)^3 = cx^5$.

Ex. 9. NT and NP are the subtangent and ordinate of a curve of which the vertex is A, and $\tan TPA = m \tan APN$, find the equation to the curve.

$$(y^2 + x^2)^{m+1} = c^{2m} x^2.$$

Ex. 10. $\int_x (xy) = \frac{x}{n} \int_x y$; $\therefore y^{n-1} \cdot x^{n-2} = a^{2n-3}$.

Ex. 11. $\int_x (xy^2) = \frac{2}{3} x \int_x y^2$; $\therefore y^2 = cx$.

99. The equation $(a + bx + cy)\,dx + (a_1 + b_1 x + c_1 y)\,dy = 0$ can be rendered homogeneous by making

$$v = a + bx + cy, \quad \text{and } z = a_1 + b_1 x + c_1 y;$$

$$\therefore dv = bdx + cdy, \quad dz = b_1 dx + c_1 dy;$$

$$\therefore c_1 dv - cdz = (bc_1 - b_1 c)\,dx,$$

$$bdz - b_1 dv = (bc_1 - b_1 c)\,dy;$$

whence by substitution the equation becomes

$$v(c_1dv - cdz) + z(bdz - b_1dv) = 0,$$
$$\text{or } (vc_1 - b_1z)dv + (bz - cv)\,dz = 0,$$

which is a homogeneous equation.

COR. This method is inapplicable when $bc_1 = b_1c$; but since then $c_1 = \frac{b_1c}{b}$, the equation becomes

$$(a + bx + cy)\,dx + \left(a_1 + b_1x + b_1\frac{cy}{b}\right)dy = 0,$$

$$\text{i. e. } (a + bx + cy)\,dx + \{a_1 + \frac{b_1}{b}(bx + cy)\}\,dy = 0,$$

an equation in which the variables may be separated by making $bx + cy = z$; $\therefore\ dx = \frac{dz - cdy}{b}$;

$$\therefore\ (a + z)\frac{dz - cdy}{b} + \left(a_1 + \frac{b_1z}{b}\right)dy = 0;$$

$$\therefore\ (a + z)\,dz - (ca + cz - a_1b - b_1z)\,dy = 0;$$

$$\therefore\ \frac{dy}{dz} = \frac{(a + z)}{ca - a_1b + (c - b_1)z} = \frac{(a + z)}{\alpha + \beta z},$$

where $\alpha = ca - a_1b$ and $\beta = c - b_1$, the integral of which may be readily found.

100. To integrate the *linear* equation, (so called since the first power of y is alone involved),

$$\frac{dy}{dx} + Py = Q \ldots\ldots(1),$$

in which P and Q are functions of x.

$$\text{Since } \frac{d}{dx}ye^{\int_x P} = \frac{dy}{dx}e^{\int_x P} + e^{\int_x P}\,.\,Py$$
$$= e^{\int_x P}\left\{\frac{dy}{dx} + Py\right\}.$$

It is obvious that if both sides of (1) be multiplied by $e^{\int_x P}$, the left hand will be a complete differential, and the right hand a function of x alone; multiply therefore by $e^{\int_x P}$;

$$\therefore\ e^{\int_x P}\left\{\frac{dy}{dx} + Py\right\} = e^{\int_x P}\,.\,Q;\ \text{ or } \frac{d}{dx}\left(ye^{\int_x P}\right) = e^{\int_x P}\,.\,Q;$$

$$\therefore \text{ integrating } ye^{\int_x P} = C + \int e^{\int_x P}\,.\,Q;$$
$$\text{or } y = Ce^{-\int_x P} + e^{-\int_x P}\int e^{\int_x P}\,.\,Q.$$

Ex. 1. Let $\dfrac{dy}{dx} + y = ax^3$.

Here $P = 1$, $\int_x P = x$; $\therefore\ e^{\int_x P} = e^x$, $Q = ax^3$;

$$\therefore\ ye^x = C + a\int_x e^x . x^3 = C + ae^x\{x^3 - 3x^2 + 6x - 6\};$$

$$\therefore\ y = Ce^{-x} + a\{x^3 - 3x^2 + 6x - 6\}.$$

Ex. 2. $(1 + x^2)\dfrac{dy}{dx} - yx = a$; or $\dfrac{dy}{dx} - y\dfrac{x}{1+x^2} = \dfrac{a}{1+x^2}$.

Here $P = -\dfrac{x}{1+x^2}$; $\int_x P = \log\dfrac{1}{\sqrt{1+x^2}}$; $e^{\int_x P} = \dfrac{1}{\sqrt{1+x^2}}$;

$$\therefore\ y\frac{1}{\sqrt{1+x^2}} = a\int_x \frac{1}{\sqrt{1+x^2}} \times \frac{1}{1+x^2}$$

$$= a\int_x \frac{1}{(1+x^2)^{\frac{3}{2}}} = \frac{ax}{\sqrt{1+x^2}} + c;$$

$$\therefore\ y = ax + c\sqrt{1+x^2}.$$

Ex. 3. $\dfrac{dy}{dx} = a + bx + cy$; $\therefore\ Ce^{cx} = b + c(a + bx + cy)$.

Ex. 4. $\dfrac{dy}{dx} + \dfrac{ny}{x} = \dfrac{a}{x^n}$; $\therefore\ yx^n = a(x + b)$.

101. The equation $y^{m-1}\dfrac{dy}{dx} + Py^m = Qy^n$ may be reduced to the preceding form, in the following manner.

Divide by y^n; $\therefore\ y^{m-n-1}\dfrac{dy}{dx} + Py^{m-n} = Q$.

Let $y^{m-n} = (m-n)z$; $\therefore\ y^{m-n-1}\dfrac{dy}{dx} = \dfrac{dz}{dx}$;

$$\therefore\ \frac{dz}{dx} + (m-n)Pz = Q.$$

Ex. 1. $v\dfrac{dv}{ds} - \dfrac{hv^2}{s} = -\dfrac{m}{s^2}$.

Let $v^2 = 2z$; $\therefore\ v\dfrac{dv}{ds} = \dfrac{dz}{ds}$;

$$\therefore\ \frac{dz}{ds} - \frac{2hz}{s} = -\frac{m}{s^2}.$$

Here $P = -\dfrac{2h}{s}$; $\therefore\ \int_s P = -2h\log(s) = \log\dfrac{1}{s^{2h}}$; $\therefore\ e^{\int_s P} = \dfrac{1}{s^{2h}}$;

$$\therefore\ zs^{-2h} = -m\int_s s^{-(2h+2)} = c + \frac{ms^{-(2h+1)}}{2h+1};$$

$$\therefore\ z = \frac{v^2}{2} = cs^{2h} + \frac{m}{(2h+1)s}.$$

Ex. 2. $\frac{dy}{dx} + y = xy^3$; $\therefore\ \frac{1}{y^2} = x + \frac{1}{2} + Ce^{2x}$.

Ex. 3. $\frac{dy}{dx} + \frac{xy}{1-x^2} = x\sqrt{y}$; $\therefore\ \sqrt{y} = C\sqrt[4]{1-x^2} - \frac{1}{3}(1-x^2)$.

Ex. 4. $xy^2dy + y^3dx = \frac{adx}{x}$;

$$\therefore\ y^3 = \frac{3a}{2x} + \frac{C}{x^3}.$$

Integration of exact Differentials. The method of finding a factor which will render a function integrable.

102. The equation $Mdx + Ndy = 0$ is not always the result of the differentiation of $f(xy) = c$; for after the differentiation, its terms may have been divided by a common factor, or the equation may arise from the elimination of an arbitrary constant between the primitive equation and its derivative.

But whenever $Mdx + Ndy = 0$ is the complete differential of a function of two variables, the condition $\frac{d^2u}{dxdy} = \frac{d^2u}{dydx}$ is fulfilled, or since $M = \frac{du}{dx}$ and $N = \frac{du}{dy}$;

$$\therefore\ \frac{dM}{dy} = \frac{d^2u}{dxdy} = \frac{dN}{dx}.$$

Hence, we may readily find whether any equation of the form $Mdx + Ndy = 0$, be a complete differential; and if it be, then we can by integrating the partial differential equations find u; for since $M = \frac{du}{dx}$, M is the partial differential coefficient of u, with regard to x, considering x alone to

vary, and its integral will give all the terms in which x is to be found: let the integration be performed. Then

$$u = \int_x M + Y.$$

Here instead of adding a constant C, we put Y, for as y has been supposed not to vary, the constant will include those terms of the original equation which are functions of y alone. Next to determine Y: differentiate with regard to y;

$$\therefore \frac{du}{dy} = \frac{d\int_x M}{dy} + \frac{dY}{dy}.$$

But $\frac{du}{dy} = N$; $\therefore \frac{dY}{dy} = N - \frac{d\int_x M}{dy}$;

$$\therefore Y = \int_y \left(N - \frac{d\int_x M}{dy}\right) + C.$$

$$\therefore u = \int_x M + \int_y \left(N - \frac{d\int_x M}{dy}\right) + C.$$

103. Since Y ought to be a function of y only, $\int_y \left(N - \frac{d\int_x M}{dy}\right)$ should be independent of x.

To prove this, let $y + \delta y$ be put for y in $\int_x M$;

$$\therefore \int_x \left(M + \frac{dM}{dy}\delta y + \&c.\right) = \int_x M + \delta y \int_x \frac{dM}{dy} + \&c.$$

Hence $\frac{d\int_x M}{dy} = \int_x \frac{dM}{dy}$; $\therefore Y = \int_y \left(N - \int_x \frac{dM}{dy}\right) + C$;

$$\therefore \frac{dY}{dy} = N - \int_x \frac{dM}{dy}; \quad \therefore \frac{d^2 Y}{dx dy} = \frac{dN}{dx} - \frac{dM}{dy} = 0;$$

or since $\frac{dY}{dy}$ differentiated with regard to x vanishes; Y is a function of y only: the same result would have been obtained by integrating N or $\frac{du}{dy}$ in the first instance.

Ex. 1. Let $du = \frac{2dx}{\sqrt{x^2 - y^2}} - \frac{2x dy}{y\sqrt{x^2 - y^2}}$.

Here $M = \frac{2}{\sqrt{x^2 - y^2}}$; $N = \frac{-2x}{y\sqrt{x^2 - y^2}}$

$$\frac{dM}{dy} = \frac{2y}{(x^2 - y^2)^{\frac{3}{2}}}, \quad \frac{dN}{dx} = \frac{-2}{y}\left(\frac{-y^2}{(x^2 - y^2)^{\frac{3}{2}}}\right) = \frac{2y}{(x^2 - y^2)^{\frac{3}{2}}};$$

$$\therefore\ u = \int_x M + Y = 2\log\left(x + \sqrt{x^2 - y^2}\right) + Y,$$

$$\frac{du}{dy} = \frac{-2y}{\left(x + \sqrt{x^2 - y^2}\right)\sqrt{x^2 - y^2}} + \frac{dY}{dy} = \frac{-2x}{y\sqrt{x^2 - y^2}};$$

$$\therefore\ \frac{dY}{dy} = \frac{2}{\sqrt{x^2 - y^2}}\left\{\frac{y}{x + \sqrt{x^2 - y^2}} - \frac{x}{y}\right\}$$

$$= \frac{-2}{\sqrt{x^2 - y^2}}\left\{\frac{x^2 - y^2 + x\sqrt{x^2 - y^2}}{y\left(x + \sqrt{x^2 - y^2}\right)}\right\} = \frac{-2}{y};$$

$$\therefore\ Y = C - 2\log y;$$

$$\therefore\ u = \log\left(\frac{x + \sqrt{x^2 - y^2}}{y}\right)^2 + C.$$

Ex. 2. Let $du = \dfrac{a(xdx + ydy)}{\sqrt{x^2 + y^2}} + \dfrac{ydx - xdy}{x^2 + y^2} + 3by^2dy = 0.$

$$M = \frac{ax}{\sqrt{x^2 + y^2}} + \frac{y}{x^2 + y^2};\quad N = \frac{ay}{\sqrt{x^2 + y^2}} - \frac{x}{x^2 + y^2} + 3by^2.$$

Here $\dfrac{dM}{dy} = \dfrac{-ay}{(x^2 + y^2)^{\frac{3}{2}}} + \dfrac{x^2 - y^2}{(x^2 + y^2)^2} = \dfrac{dN}{dx};$

$$\therefore\ u = \int_x M + Y = a\sqrt{x^2 + y^2} + \tan^{-1}\frac{x}{y} + Y,$$

$$\frac{du}{dy} = \frac{ay}{\sqrt{x^2 + y^2}} - \frac{x}{y^2 + x^2} + \frac{dY}{dy};\quad \therefore\ \frac{dY}{dy} = 3by^2;$$

$\therefore\ Y = by^3 + C$, and $u = a\sqrt{x^2 + y^2} + \tan^{-1}\dfrac{x}{y} + by^3 + C.$

Ex. 3. $\displaystyle\int \frac{xdy - ydx}{x^2 + y^2} = \tan^{-1}\frac{x}{y} + C.$

Ex. 4. $\dfrac{dx}{x} + \dfrac{y^2dx}{x^3} - \dfrac{ydy}{x^2} + \dfrac{dy}{2y} + \dfrac{ydx - xdy}{x^3}\sqrt{x^2 + y^2} = 0;$

$$\therefore \log(xy) - \frac{y}{x^2}\left(y + \sqrt{x^2 + y^2}\right) + \log\left(y\sqrt{x^2 + y^2} - y^2\right) = C.$$

104. When the equation $Mdx + Ndy = 0$ does not fulfil the *criterion of integrability*, $\dfrac{dM}{dy} = \dfrac{dN}{dx}$, it is no longer a complete differential, some factor having disappeared from it. Could however the factor be restored, every equation of this class might be integrated by the same process: but

there is great difficulty in finding this factor; in most cases the differential equation, by which it is to be determined, is more complicated than the original one.

Thus, suppose z to be the factor, then $Mzdx + Nzdy = 0$ is a complete differential, and therefore

$$\frac{d(Mz)}{dy} = \frac{d(Nz)}{dx};$$

$$\therefore\ z\frac{dM}{dy} + M\frac{dz}{dy} = z\frac{dN}{dx} + N\frac{dz}{dx};$$

whence z is to be found, a problem seldom practicable.

105. When, however, any factor of the equation

$$Mdx + Ndy = 0$$

is known, an infinite number of factors may be found which will render the equation integrable; for let z be a factor,

$$\therefore\ du = zMdx + Nzdy;$$

$$\therefore\ \phi(u)\,du = z\phi(u)\,Mdx + Nz\phi(u)\,dy;$$

and since the first member of the equation is an exact differential, the second member is also; $\therefore$ z multiplied by any function of (u), will make the equation integrable.

We may sometimes find the factor when the differential equation can be divided into two parts, for each of which a factor can be found; for let the equation be

$$pdx + qdy + p_1dx + q_1dy = 0 \quad (1),$$

and suppose that z and z_1 are the factors which will make $pdx + qdy$ and $p_1dx + q_1dy$ integrable, so that

$$zpdx + zqdy = du; \text{ and } z_1p_1dx + z_1q_1dy = du_1;$$

$\therefore$ $z\phi(u)$ and $z_1\phi(u_1)$ will include all the factors which will render the two equations separately integrable; if therefore we can make $z\phi(u) = z_1\phi(u_1)$, we shall obtain a factor which will render the equation (1) integrable.

Ex. 1. Let $\dfrac{adx}{x} + \dfrac{bdy}{y} = \dfrac{cx^m dx}{y^b}$.

$$\because\ \frac{adx}{x} + \frac{bdy}{y} = d.\log(x^a y^b); \quad \therefore\ z = 1; \quad u = x^a.y^b;$$

$$\frac{cx^m dx}{y^b} \text{ is integrable if } z_1 = y^b; \quad \therefore\ u_1 = \frac{cx^{m+1}}{m+1};$$

$$\therefore\ z\phi(u) = \phi(x^a y^b); \quad z_1\phi(u_1) = y^b\phi(x^{m+1})$$

Let $\phi(x^a y^b) = x^{ka} y^{kb}$; $\phi(x^{m+1}) = x^{k_1 m+1}$;

$\therefore\ x^{ka} y^{kb} = y^b x^{k_1 m+1}$; $\therefore\ kb = b$; $\therefore\ k = 1$; $k_1 = \dfrac{a-1}{m}$;

$\therefore$ the factor is $x^a y^b$, whence integrating

$$x^a y^b = \frac{cx^{m+a+1}}{m+a+1} + C.$$

Ex. 2. $xdy - 2ydx = adx$. Here

$$z = \frac{1}{xy};\quad u = \log\left(\frac{y}{x^2}\right);\quad z_1 = 1;\quad u_1 = ax;$$

$$\therefore\ \frac{1}{x^3} \text{ is the factor; and } y + cx^2 = \frac{a}{2}.$$

Ex. 3. $aydx + bxdy = x^m y^n (a_1 ydx + b_1 xdy)$.

The factor is $x^{ka-1} y^{kb-1}$; $k = \dfrac{a_1 n - b_1 m}{ab_1 - a_1 b}$:

the integral of this equation might also be found by making $x^a y^b = z$; $x^{a_1} y^{b_1} = v$, from which we shall obtain

$$\frac{z^k}{a_1 n - b_1 m} + \frac{v^{k_1}}{an - bm} = C;\ \text{where } k_1 = \frac{an - bm}{ab_1 - a_1 b}.$$

106. The factor may be determined when z contains only one variable as x, for then $\dfrac{dz}{dy} = 0$, and therefore

$$\frac{dz}{zdx} = \frac{1}{N}\left(\frac{dM}{dy} - \frac{dN}{dx}\right).$$

The right-hand side must be a function of x only, which is the case in the linear equation, for $N = 1$, and M contains only the first power of y; therefore integrating,

$$\log\frac{z}{c} = X;\quad \therefore\ z = ce^X.$$

But to find *a priori* the multiplier which will make the equation $dy + (Py - Q)dx = 0$ an exact differential.

Let z be the multiplier: multiply by it;

$$\therefore\ zdy + z(Py - Q)dx = Ndy + Mdx;$$

$$\therefore\ \frac{dN}{dx} = \frac{dz}{dx};\quad \frac{dM}{dy} = (Py - Q)\frac{dz}{dy} + Pz;$$

$$\therefore\ \frac{dz}{dx} = (Py - Q)\frac{dz}{dy} + Pz;$$

$$\therefore \frac{dz}{dx} dx = (Py - Q) dx \frac{dz}{dy} + Pzdx$$

$$= -\frac{dz}{dy} dy + Pzdx; \text{ since } (Py - Q) dx = -dy;$$

$$\therefore \frac{dz}{dx} dx + \frac{dz}{dy} dy = dz = Pzdx;$$

$$\text{i.e. } \frac{1}{z} \frac{dz}{dx} = P; \quad \therefore z = e^{\int_x P};$$

which justifies the assumption made in article (100).

107. The factor may also be found when the equation is homogeneous.

For let $M + N\dfrac{dy}{dx} = 0$ be the differential equation, supposed to be homogeneous and of m dimensions, and let z be the factor, a homogeneous function of the n^{th} degree;

$$\therefore zMdx + zNdy = du \dots\dots\dots (1).$$

Hence, since u must be of $m + n + 1$ dimensions,

$$zMx + zNy = (m + n + 1)u \dots\dots\dots (2),$$

Art. (112), Diff. Calc.; therefore dividing (1) by (2),

$$\frac{Mdx + Ndy}{Mx + Ny} = \frac{1}{m + n + 1} \cdot \frac{du}{u};$$

and since the right-hand side of the equation is a complete differential, the left-hand must be so also; and $\therefore \dfrac{1}{Mx + Ny}$ is the factor required.

Ex. Let $ydy + (x - ny)dx = 0$;

$$\therefore \text{the factor} = \frac{1}{x^2 - nyx + y};$$

$$\therefore Mz = \frac{x - ny}{x^2 - nyx + y^2}, \quad Nz = \frac{y}{x^2 - nyx + y^2},$$

$$\text{and } \frac{d(Mz)}{dy} = \frac{ny^2 - 2xy}{(x^2 - nyx + y^2)^2} = \frac{d(Nz)}{dx}.$$

108. We shall now add a few problems illustrating the solution of differential equations.

Find the curve which cuts any number of curves of a given species at a given angle.

Let y and x be the co-ordinates of the curve of given species,

y_1 and x_1 those of the required curve,

m = tangent of given angle.

$$\text{Then } \tan^{-1} m = \tan^{-1} \frac{dy}{dx} - \tan^{-1} \frac{dy_1}{dx_1};$$

$$\therefore\ m = \frac{\dfrac{dy}{dx} - \dfrac{dy_1}{dx_1}}{1 + \dfrac{dy}{dx} \cdot \dfrac{dy_1}{dx_1}},$$

and $\dfrac{dy}{dx}$ may be found from the given curve, and is a function of x and y, or $\phi(xy)$, and since at the point of intersection the co-ordinates of both curves are the same, we may for x_1 and y_1 put x and y; and then the equation to the required curve is

$$m \left\{ 1 + \phi(xy) \frac{dy}{dx} \right\} = \phi(xy) - \frac{dy}{dx},$$

which is of the first order and degree.

COR. If the required curve cut the given curves at right angles,

$$\text{then } m = \frac{1}{0}; \quad \therefore\ 1 + \phi(x, y) \frac{dy}{dx} = 0; \quad \therefore\ \frac{dy}{dx} = -\frac{1}{\phi(x, y)},$$

which is the equation to the *Orthogonal Trajectory.*

Ex. 1. Find the curve which will cut all the parabolas that have a common vertex and axis at right angles.

Let $y^2 = 2mx$ be the equation to one of the parabolas;

$$\therefore\ \phi(xy) = \frac{m}{y} = \frac{y}{2x};$$

$$\therefore\ \frac{dy}{dx} = -\frac{2x}{y}; \quad \therefore\ \frac{y^2}{2} = (c^2 - x^2),$$

the equation to an ellipse of which the centre is the common vertex of the parabolas, and the major axis is perpendicular to the common axis, the ratio of the axes being $\sqrt{2} : 1$; c being indeterminate shews that any ellipse of which the axes are in the given ratio will cut the parabolas at right angles.

Ex. 2. Find the curve which will cut at right angles all the ellipses that have a common centre, coincident major axes, and the ratio of their axes constant.

Let $y^2 = n(a^2 - x^2)$ be the equation to one of the ellipses in which $\sqrt{n} = \frac{b}{a}$;

$$\therefore \frac{dy}{dx} = -n\frac{x}{y} = -\frac{dx}{dy};$$

$$\therefore n\frac{dy}{y} = \frac{dx}{x}; \quad \therefore n\log\left(\frac{y}{b}\right) = \log\left(\frac{x}{c}\right); \quad \therefore y^n = \frac{b^n}{c}x,$$

the equation to a parabola, of which the vertex is in the common centre of the ellipses.

If $n = 2$, $y^2 = \frac{b^2}{c}x$, the common parabola: this case is obviously the converse of the preceding problem.

Ex. 3. Find the curve which intersects at an angle of 45^0, all the straight lines drawn from the origin to meet it.

Let $y = ax$ be one of the lines;

$$\therefore \phi(xy) = a = \frac{y}{x}; \quad m = 1; \quad \therefore 1 + \frac{y}{x}\frac{dy}{dx} = \frac{y}{x} - \frac{dy}{dx},$$

a homogeneous equation,

$$\text{whence } \log\left(\sqrt{\frac{x^2+y^2}{c}}\right) = -\tan^{-1}\left(\frac{y}{x}\right).$$

$$\text{Let } y = r\sin\theta, \quad x = r\cos(\pi - \theta) = -r\cos\theta;$$

$$\therefore r = \sqrt{x^2+y^2}, \quad \frac{y}{x} = -\tan\theta; \quad \therefore \log\left(\frac{r}{c}\right) = \theta; \quad r = ce^\theta,$$

the equation to the logarithmic spiral.

Ex. 4. The orthogonal trajectory of $y^2 - 4ax + 2x^2 = 0$ is $y^2 = m(a - x)$.

109. To integrate *Riccati's* equation, so called from its proposer,

$$\frac{dy}{dx} + by^2 = ax^m.$$

(1) If $m = 0$, then $\frac{dy}{dx} = a - by^2$, which is easily integrable.

(2) If m be not $= 0$, we must proceed as follows.

Case 1. Let $y = \frac{1}{bx} + \frac{z}{x^2}$; $\therefore \frac{dy}{dx} = -\frac{1}{bx^2} + \frac{dz}{dx}\cdot\frac{1}{x^2} - \frac{2z}{x^3}$,

$$by^2 = \frac{1}{bx^2} + \frac{bz^2}{x^4} + \frac{2z}{x^3};$$

$$\therefore \frac{dy}{dx}+by^2=\frac{dz}{dx}\frac{1}{x^2}+\frac{bz^2}{x^4}=ax^m;$$

$$\therefore \frac{dz}{dx}+\frac{bz^2}{x^2}=ax^{m+2},$$

which is homogeneous if $m+2=0$, or $m=-2$, and the variables may be separated if $m=-4$; for then

$$\frac{dz}{dx}+\frac{bz^2}{x^2}=\frac{a}{x^2};\quad \therefore \frac{dz}{bz^2-a}+\frac{dx}{x^2}=0.$$

If m have any other value, make

$$z=\frac{1}{y_1};\quad x^{m+3}=x_1;\quad \therefore dz=-\frac{dy_1}{y_1^2};\quad x^{m+2}\,dx=\frac{dx_1}{m+3}.$$

$$\text{Also } \frac{1}{x}=x_1^{-\frac{1}{m+3}};\quad \therefore \frac{dx}{x^2}=\frac{dx_1}{m+3}x_1^{-\frac{m+4}{m+3}},$$

$$\text{whence } -\frac{dy_1}{y_1^2}+\frac{b}{(m+3)y_1^2}x_1^{-\frac{m+4}{m+3}}dx_1=\frac{a}{m+3}dx_1.$$

$$\text{Let } \frac{b}{m+3}=a_1;\quad \frac{a}{m+3}=b_1;\quad -\frac{m+4}{m+3}=m_1.$$

$$\text{Then } dy_1+b_1y_1^2dx_1=a_1x_1^{m_1}dx_1,$$

which is of the same form as the original equation, and may be made homogeneous if $m_1=-2$, and the variables may be separated by the preceding process if $m_1=-4$.

By continuing the same methods it is evident that we shall have a similar equation, $dy_2+b_2y_2^2dx_2=a_2x_2^{m_2}dx_2$, where $m_2=-\frac{m_1+4}{m_1+3}$; and b_2, a_2 are derived from b_1 and a as these were from b and a; which equation will be integrable if $m_2=-4$.

And hence if among the series of indices

$$-m,\ -\frac{m+4}{m+3},\quad -\frac{m_1+4}{m_1+3},\quad -\frac{m_2+4}{m_2+3},\ \text{\&c.}$$

any one $=-4$, the equation is integrable. And by successively putting these indices $=-4$, we find the values of m to be, -4, $-\frac{8}{3}$, $-\frac{12}{5}$, $-\frac{16}{7}$, &c., which are included under the form $-\frac{4n}{2n-1}$, n being any integer.

CASE 2. Make in the original equation $y=\frac{1}{y_1}$;

$$\therefore -\frac{dy_1}{y_1^2}+\frac{b}{y_1^2}dx=ax^m dx; \quad \therefore dy_1+ay_1^2x^m dx=bdx.$$

Let $x^{m+1}=x_1$; $\therefore x=x_1^{\frac{1}{m+1}}$; $\therefore dx=\frac{1}{m+1}x_1^{-\frac{m}{m+1}}dx_1$.

$$\text{And } dy_1+\frac{a}{m+1}y_1^2dx_1=\frac{b}{m+1}x_1^{-\frac{m}{m+1}}dx_1;$$

or putting $\frac{a}{m+1}=b_1$, $\frac{b}{m+1}=a_1$, and $-\frac{m}{m+1}=m_1$;

$$dy_1+b_1y_1^2dx_1=a_1x_1^{m_1}dx_1,$$

which may be integrated by the former method if $m_1=\frac{-4n}{2n-1}$; or if $\frac{m}{m+1}=\frac{4n}{2n-1}$, whence $m=\frac{-4n}{2n+1}$. Hence Riccati's equation is integrable when m is of the form $\frac{-4n}{2n\mp 1}$. The first case belongs to the upper, the second to the lower sign.

Ex. 1. Integrate $dy+y^2dx=\frac{a^2dx}{x^{\frac{4}{3}}}$.

$$\text{Here } -\tfrac{4}{3} \text{ is of the form } -\frac{4n}{2n+1};$$

$$\therefore \text{ let } y=\frac{1}{y_1}, \text{ and let } x^{m+1}=x^{-\frac{4}{3}+1}=x^{-\frac{1}{3}}=x_1;$$

$$\therefore x=x_1^{-3}; \ dx=-3x_1^{-4}dx_1; \ x^{\frac{4}{3}}=x_1^{-4};$$

$$\therefore -\frac{dy_1}{y_1^2}-\frac{3}{y_1^2}x_1^{-4}dx_1=-3a^2dx_1,$$

$$dy_1-3a^2y_1^2dx_1=-3x_1^{-4}dx_1.$$

Let $-3a^2=b_1$, $-3=a_1$; $\therefore dy_1+b_1y_1^2dx_1=a_1x_1^{-4}dx_1$.

Now let $y_1=\frac{1}{b_1x_1}+\frac{z_1}{x_1^2}$.

Then $\frac{dz_1}{dx_1}+b_1z_1^2\frac{1}{x_1^2}=a_1x_1^{-2}$, or $x_1^2\frac{dz_1}{dx_1}=(a_1-b_1z^2)$;

$$\therefore \frac{1}{x_1^2}\frac{dx_1}{dz_1}=\frac{1}{a_1-b_1z_1^2}=\frac{1}{3(a^2z_1^2-1)},$$

$$\frac{3a}{x_1}=\log\sqrt{\frac{az_1+1}{az_1-1}}\times\frac{1}{c}=\log\frac{1}{c}\sqrt{\frac{3a^2x_1^2y_1+x_1+3a}{3a^2x_1^2y_1+x_1-3a}};$$

$$\therefore \text{ since } \frac{1}{x_1}=x^{\frac{1}{3}} \text{ and } y_1=\frac{1}{y};$$

$$\therefore e^{6ax^{\frac{1}{3}}} = \frac{1}{c^2}\left\{\frac{3a^2x^{-\frac{2}{3}} + y(3a + x^{-\frac{1}{3}})}{3a^2x^{-\frac{2}{3}} + y(x^{-\frac{1}{3}} - 3a)}\right\}$$

$$= \frac{1}{c^2}\left\{\frac{3a^2x^{-\frac{1}{3}} + y(1 + 3ax^{\frac{1}{3}})}{3a^2x^{-\frac{1}{3}} + y(1 - 3ax^{\frac{1}{3}})}\right\};$$

$$\therefore c^2 = C = e^{-6ax^{\frac{1}{3}}}\left\{\frac{3a^2x^{-\frac{1}{3}} + y(1 + 3ax^{\frac{1}{3}})}{3a^2x^{-\frac{1}{3}} + y(1 - 3ax^{\frac{1}{3}})}\right\}.$$

Ex. 2. Let $dy + y^2dx = \dfrac{a^2dx}{x^{\frac{8}{3}}}$.

Here $-\dfrac{8}{3}$ is of the form $\dfrac{-4n}{2n-1}$;

$\therefore$ let $y = \dfrac{1}{x} + \dfrac{z}{x^2}$; and $\because b = 1$, $m = -\dfrac{8}{3}$;

$\therefore \dfrac{dz}{dx} + \dfrac{bz^2}{x^2} = ax^{m+2}$ becomes $\dfrac{dz}{dx} + z^2\dfrac{1}{x^2} = a^2x^{-\frac{2}{3}}$.

Let $z = \dfrac{1}{y_1}$, and $\because m + 2 = -\dfrac{2}{3}$; $\therefore m + 1 = \dfrac{1}{3}$.

Let $x^{\frac{1}{3}} = x_1$; $\therefore x^{-\frac{2}{3}}dx = 3dx_1$; $\dfrac{1}{x} = \dfrac{1}{x_1^{\,3}}$; $\therefore \dfrac{dx}{x^2} = \dfrac{3dx_1}{x_1^{\,4}}$;

$\therefore \dfrac{-dy_1}{y_1^{\,2}} + \dfrac{1}{y_1^{\,2}}\dfrac{3dx_1}{x_1^{\,4}} = 3a^2dx_1$; $\therefore dy_1 + 3a^2y_1^{\,2}dx_1 = 3x_1^{-4}dx_1$,

which, as has been shewn, is integrable.

110. It sometimes happens that equations in which the variables are separated admit of algebraical integrals, although the integral of each part is transcendental.

Thus, since $\dfrac{dy}{\sqrt{1-y^2}} + \dfrac{dx}{\sqrt{1-x^2}} = 0$, (1);

$$\therefore \sin^{-1}x + \sin^{-1}y = C = \sin^{-1}c.$$

$$\therefore x\sqrt{1-y^2} + y\sqrt{1-x^2} = c;$$

we may however obtain the same result, thus

$$(1) \quad \times xy; \quad \therefore \int\frac{xydx}{\sqrt{1-x^2}} + \int\frac{xydy}{\sqrt{1-y^2}} = 0;$$

or $-y\sqrt{1-x^2} + \int dy\sqrt{1-x^2} - x\sqrt{1-y^2} + \int dx\sqrt{1-x^2} = -c$;

which since $dy\sqrt{1-x^2} + dx\sqrt{1-y^2} = 0$, reduces itself to

$$y\sqrt{1-x^2} + x\sqrt{1-y^2} = c,$$

the required algebraic relation between y and x.

111. Again, let $\dfrac{dy}{\sqrt{a+by+cy^2}}+\dfrac{dx}{\sqrt{a+bx+cx^2}}=0$;

make y and x functions of another variable t, so that

$$\frac{dy}{dt}=\sqrt{a+by+cy^2};\quad \therefore \frac{dx}{dt}=-\sqrt{a+bx+cx^2};$$

ct $x+y=p$, then squaring and differentiating,

$$2\frac{d^2y}{dt^2}=b+2cy;\quad 2\frac{d^2x}{dt^2}=b+2cx;$$

$$\therefore 2\frac{d^2y}{dt^2}+2\frac{d^2x}{dt^2}=2\frac{d^2p}{dt^2}=2b+2cp.$$

Multiply both sides by $\dfrac{dp}{dt}$, and integrate;

$$\therefore \frac{dp^2}{dt^2}=C+2bp+cp^2;\ \therefore \frac{dp}{dt}=\frac{dx}{dt}+\frac{dy}{dt}=\sqrt{C+2bp+cp^2};$$

or $\sqrt{a+bx+cx^2}-\sqrt{a+by+cy^2}=\sqrt{C+2b(x+y)+c(x+y)^2}$.

Ex. 2. $\dfrac{dy}{\sqrt{a+by+cy^2+ey^3+fy^4}}+\dfrac{dx}{\sqrt{a+bx+cx^2+ex^3+fx^4}}=0.$

Let $\dfrac{dx}{dt}=\sqrt{a+bx+cx^2+ex^3+fx^4}=\sqrt{X}$;

$$\therefore \frac{dy}{dt}=-\sqrt{a+by+cy^2+ey^3+fy^4}=\sqrt{Y}.$$

Make $x+y=p$; $x-y=q$; whence squaring and differentiating,

$$\frac{d^2x}{dt^2}+\frac{d^2y}{dt^2}=\frac{d^2p}{dt^2}=\tfrac{1}{2}\left(\frac{dX}{dx}+\frac{dY}{dy}\right)$$
$$=b+c(x+y)+\tfrac{3}{2}e(x^2+y^2)+2f(x^3+y^3)$$
$$=b+cp+\tfrac{3}{4}e(p^2+q^2)+\tfrac{1}{2}fp(p^2+3q^2),$$

$$\frac{dx^2}{dt^2}-\frac{dy^2}{dt^2}=\frac{dp\,.\,dq}{dt^2}=bq+cpq+\frac{e}{4}(3p^2q+q^3)+\frac{f}{2}qp(p^2+q^2);$$

$$\therefore q\frac{d^2p}{dt^2}-\frac{dp\,.\,dq}{dt^2}=\frac{e}{2}q^3+fpq^3;$$

$\therefore$ multiplying both sides by $\dfrac{2}{q^3}\cdot\dfrac{dp}{dt}$;

$$\frac{2}{q^2}\cdot\frac{d^2p}{dt^2}\cdot\frac{dp}{dt}-\frac{1}{q^3}\cdot\frac{dp^2}{dt^2}\cdot 2\frac{dq}{dt}=e\frac{dp}{dt}+2fp\cdot\frac{dp}{dt};$$

whence integrating

$$\frac{1}{q^2}\frac{dp^2}{dt^2} = C + ep + fp^2; \quad \therefore \quad \frac{dp}{dt} = q\sqrt{C + ep + fp^2};$$

$$\text{or} \because \frac{dp}{dt} = \frac{dx}{dt} + \frac{dy}{dt}, \quad q = x - y, \quad \text{and} \quad p = x + y,$$

$$\sqrt{a + bx + cx^2 + ex^3 + fx^4} - \sqrt{a + by + cy^2 + ey^3 + fy^4}$$
$$= (x - y)\sqrt{c + e(x + y) + f(x + y)^2},$$

an algebraic equation, which may be put also under a rational form: for writing $(x - y)\sqrt{P}$ instead of the right-hand side of the equation, inverting and multiplying by $X - Y$;

$$\therefore \frac{X - Y}{\sqrt{X} - \sqrt{Y}} = \frac{X - Y}{(x - y)\sqrt{P}};$$

$$\text{or } \sqrt{X} + \sqrt{Y} = \frac{X - Y}{(x - y)\sqrt{P}}; \quad \therefore \text{ by addition,}$$

$$2\sqrt{X} = (x - y)\sqrt{P} + \frac{X - Y}{(x - y)\sqrt{P}} = \frac{(x - y)^2 P + (X - Y)}{(x - y)\sqrt{P}};$$

and squaring both sides, the equation becomes rational.

Ex. 3. $$\frac{d\phi}{\sqrt{1 - e^2 \sin^2\phi}} + \frac{d\theta}{\sqrt{1 - e^2 \sin^2\theta}} = 0.$$

Let $\frac{d\phi}{dt} = \sqrt{1 - e^2 \sin^2\phi}$; $\therefore$ $\frac{d\theta}{dt} = -\sqrt{1 - e^2 \sin^2\theta}$;

$$\therefore \frac{d^2\phi}{dt^2} = -e^2 \sin\phi \cdot \cos\phi; \quad \frac{d^2\theta}{dt^2} = -e^2 \sin\theta \cdot \cos\theta;$$

$$\therefore \frac{d^2\phi}{dt^2} + \frac{d^2\theta}{dt^2} = -\frac{e^2}{2}(\sin 2\phi + \sin 2\theta),$$

$$\frac{d^2\phi}{dt^2} - \frac{d^2\theta}{dt^2} = -\frac{e^2}{2}(\sin 2\phi - \sin 2\theta);$$

$\therefore$ by making $p = \phi + \theta$, $q = \phi - \theta$,

$$\frac{d^2p}{dt^2} = -\frac{e^2}{2} \cdot \{\sin(p + q) + \sin(p - q)\} = -e^2 \sin p \cdot \cos q,$$

$$\frac{d^2q}{dt^2} = -\frac{e^2}{2} \cdot \{\sin(p + q) - \sin(p - q)\} = -e^2 \cdot \sin q \cdot \cos p.$$

And $\frac{dp}{dt}\frac{dq}{dt} = \frac{d\phi^2}{dt^2} - \frac{d\theta^2}{dt^2} = -e^2(\sin^2\phi - \sin^2\theta)$

$$=\frac{e^2}{2}\cdot(\cos 2\phi-\cos 2\theta)=\frac{e^2}{2}\cdot\{\cos(p+q)-\cos(p-q)\}$$

$$=-e^2\sin p\sin q;$$

$$\therefore\ \frac{d^2p}{dt^2}\div\frac{dp}{dt}\cdot\frac{dq}{dt}=\frac{\cos q}{\sin q},\quad \frac{d^2q}{dt^2}\div\frac{dp}{dt}\frac{dq}{dt}=\frac{\cos p}{\sin p};$$

$$\therefore\ \frac{d\cdot\left(\frac{dp}{dt}\right)}{\frac{dp}{dt}}=\frac{\cos q}{\sin q}\cdot\frac{dq}{dt},\quad \frac{d\cdot\left(\frac{dq}{dt}\right)}{\frac{dq}{dt}}=\frac{\cos p}{\sin p}\cdot\frac{dp}{dt};$$

$$\therefore\ \log\left(\frac{dp}{dt}\right)=\log\sin q+c'=\log\sin q+\log a=\log(a\sin q);$$

$$\therefore\ \frac{dp}{dt}=a\sin q;\quad\text{also}\quad \frac{dq}{dt}=a'\sin p;$$

$$\therefore\ \sqrt{1-e^2\sin^2\phi}-\sqrt{1-e^2\sin^2\theta}=a\sin(\phi-\theta);$$

$$\text{and}\ \sqrt{1-e^2\sin^2\phi}+\sqrt{1-e^2\sin^2\theta}=a'\,.\sin(\phi+\theta).$$

COR. 1. The constants a and a' have a mutual dependence;

$$\text{for}\ \because\ \frac{dp}{dt}\cdot\frac{dq}{dt}=-e^2\sin p\cos p=aa'\sin p\,.\cos p;$$

$$\therefore\ aa'=-e^2.$$

COR. 2. The preceding equation may be put under a simple form;

$$\text{for}\ \because\ \frac{dp}{dq}=\frac{a\sin q}{a'\sin p};\quad\therefore\ a\cos q=a'\cos p+a''\ldots\ldots(1).$$

But a, a', a'' are reducible to one constant; for if μ be the value of ϕ when $\theta=0$,

$$a=\frac{-1+\sqrt{1-e^2\sin^2\mu}}{\sin\mu},\quad a'=\frac{1+\sqrt{1-e^2\sin^2\mu}}{\sin\mu},$$

$$a''=a\cos q-a'\cos p=(a-a')\cos\mu=-\frac{2\cos\mu}{\sin\mu}.$$

Substituting in (1) for a, a', a'', we have

$$\cos(\phi-\theta)\{-1-\sqrt{1-e^2\sin^2\mu}\}$$

$$=\cos(\phi+\theta)\{1+\sqrt{1-e^2\sin^2\mu}\}-2\cos\mu;$$

$$\therefore\ \cos(\phi-\theta)+\cos(\phi+\theta)$$

$$+\{\cos(\phi+\theta)-\cos(\phi-\theta)\}\sqrt{1-e^2\sin^2\mu}=2\cos\mu;$$

$$\text{or}\ \cos\phi\,.\cos\theta-\sin\phi\,.\sin\theta\sqrt{1-e^2\sin^2\mu}=\cos\mu.$$

COR. 3. If $\int \frac{d\phi}{\sqrt{1-e^2\sin^2\phi}} = f(\phi)$,

$$\text{and } \therefore \int \frac{d\theta}{\sqrt{1-e^2\sin^2\theta}} = f(\theta);$$

$$\therefore f(\phi) + f(\theta) = \text{a constant} = \beta.$$

But if $\phi = \mu$, $\theta = 0$, and $f(\theta) = 0$; $\therefore \beta = f(\mu)$;

$$\therefore f(\phi) + f(\theta) = f(\mu).$$

Integration of Differential Equations of the first Order and of the n^{th} Degree.

112. Let $\left(\frac{dy}{dx}\right)^n + P\left(\frac{dy}{dx}\right)^{n-1} + Q\left(\frac{dy}{dx}\right)^{n-2} + \&c. + U = 0$

be the equation; P, Q, &c. and U, being rational functions of x and y.

Let the equation be solved with regard to $\frac{dy}{dx}$; and let X_1, X_2, X_3, &c. be the values of $\frac{dy}{dx}$, or p thus found; then each of the equations $p = X_1$, $p = X_2$, $p = X_3$, &c. when integrated will satisfy the proposed equation, as also will the equation formed of the product of all these integrals.

Since the differential equation arises from eliminating a single constant*, raised to the n^{th} power, from the primitive equation; and since each simple integral introduces a constant, the solution will contain n constants, and therefore be more general than that from which it is derived. But if we consider that the constants are arbitrary, we may make each, equal to the constant belonging to the primitive equation, and then the result will be of the required form.

Ex. 1. Let $\frac{dy^2}{dx^2} = a^2$; $\therefore \frac{dy}{dx} = a$, and $\frac{dy}{dx} = -a$;

$$\therefore y = ax + c, \text{ and } y = -ax + c',$$

either of which satisfies the equation. Also their product

$$(y - ax - c)(y + ax - c') = 0$$

will satisfy it.

* For suppose $y - cx + c^2 = 0$; $\therefore p = c$; $\therefore y - px + p^2 = 0$, an equation of the first order and of the second degree.

For differentiating we obtain

$$\left(\frac{dy}{dx}-a\right)(y+ax-c')+\left(\frac{dy}{dx}+a\right)(y-ax-c)=0,$$

and making successively $y=ax+c$, and $y=-ax+c'$, we get the results $\frac{dy}{dx}=a$; $\frac{dy}{dx}=-a$; as we ought.

Again from the original equation, since $\frac{dy}{dx}=\pm a$; $\therefore$ $y-c=\pm ax$, and squaring both sides, $(y-c)^2=a^2x^2$.

This equation gives two lines, inclined at different directions to the axis of x, but both cutting the axis of y in the same point; and by giving to (c) different values, we may have groups of such lines in pairs. And the integral $(y-ax+c)(y+ax-c')$ gives the same result, except that each factor represents only lines inclined in the same direction; but by giving to c and c' all possible values, and taking care to collect together those straight lines in which c and c' are equal, we shall find the solutions comprised in the equation $(y-c)^2=a^2x^2$, which is limited to the single constant c.

Ex. 2. Let $\frac{dy^2}{dx^2}=ax$, or $p=\pm\sqrt{ax}$;

$$\therefore\ \frac{dy}{dx}=\sqrt{ax},\ \text{and}\ \frac{dy}{dx}=-\sqrt{ax};$$

$$\therefore\ y=\frac{2}{3}\sqrt{a}x^{\frac{3}{2}}+c,\ \text{and}\ y=-\frac{2}{3}\sqrt{a}x^{\frac{3}{2}}+c',$$

each of which is comprised in $(y-c)^2=\frac{4}{9}ax^3$.

Ex. 3. Find the curve when $s=ax+by$.

$$\text{Here}\ \frac{ds}{dx}=\sqrt{1+\frac{dy^2}{dx^2}}=a+b\frac{dy}{dx}.$$

And $\because$ $\frac{dy}{dx}$ is obviously constant, let $\frac{dy}{dx}=m$;

$\therefore$ $y=mx+c$, the equation to a straight line;

$$\therefore\ \frac{y-c}{x}=m,\ \text{and}\ \sqrt{1+\left(\frac{y-c}{x}\right)^2}=a+b\left(\frac{y-c}{x}\right).$$

Ex. 4. $p^2y+2px=y$; $y^2=2ax+a^2$.

113. When the equation only involves x and p, and can be solved with regard to x, we proceed thus:

$$\text{Since } x = f(p) = P, \text{ and } \frac{dy}{dx} = p;$$

$$\therefore \; y = px - \textstyle\int_p x = pP - \int_p P;$$

whence y is a function of p, and therefore of x.

Ex. 1. Let $x + ap = b\sqrt{1+p^2}$;

$$\therefore \; y = -ap^2 + bp\sqrt{1+p^2} - \textstyle\int_p(-ap + b\sqrt{1+p^2})$$

$$= -\frac{ap^2}{2} + \frac{bp}{2}\sqrt{1+p^2} - \frac{b}{2}\log_e(p+\sqrt{1+p^2}) + c.$$

The elimination of p will give y in terms of x.

Ex. 2. Let $(1+p^2)x = 1$; $\therefore \; x = \frac{1}{1+p^2}$, and $p = \sqrt{\frac{1-x}{x}}$;

$$\therefore \; y = px - \int_p \frac{1}{1+p^2} = px - \tan^{-1} p + C$$

$$= \sqrt{x - x^2} - \tan^{-1}\sqrt{\frac{1-x}{x}} + C.$$

Ex. 3. Let $x\frac{dy}{dx} = \sqrt{1 + \frac{dy^2}{dx^2}}$;

$$\therefore \; px = \sqrt{1+p^2}; \quad \therefore \; x = \frac{\sqrt{1+p^2}}{p};$$

$$\therefore \; y = px - \textstyle\int_p x = px - \displaystyle\int_p \frac{\sqrt{1+p^2}}{p}$$

$$= px - \int_p \frac{1}{p\sqrt{1+p^2}} - \int_p \frac{p}{\sqrt{1+p^2}}$$

$$= px - \log\left(\frac{cp}{1+\sqrt{1+p^2}}\right) - \sqrt{1+p^2};$$

$$\therefore \; y = \log\left(\frac{1+\sqrt{1+p^2}}{cp}\right) = \log\left(\frac{x+\sqrt{x^2-1}}{c}\right).$$

Ex. 4. $y = a\sqrt{1+p^2}$; $x + c = a\log(y + \sqrt{y^2 - a^2})$.

114. When the differential equation contains y, x and p, and is homogeneous with respect to y and x, the variables can be separated by making $y = zx$, or $x = zy$; $\therefore$ $z = f(p)$; for if

$$y = xz; \; \therefore \; p - z = x\frac{dz}{dx}; \quad \therefore \; \frac{1}{x}\frac{dx}{dz} = \frac{1}{p-z};$$

$$= \frac{1}{f^{-1}(z) - z}.$$

And x being found $= \phi(z)$, y may be determined in terms of z and therefore in terms of x.

Ex. 1. Let $y - px = x\sqrt{1+p^2}$.

Make $y = xz$; $\therefore\ z - p = \sqrt{1+p^2}$;

$$\therefore\ z^2 - 2zp = 1, \text{ or } p = \frac{z}{2} - \frac{1}{2z} = z + x\frac{dz}{dx};$$

$$\therefore\ x\frac{dz}{dx} = -\frac{z}{2} - \frac{1}{2z} = -\frac{1}{2}.\frac{z^2+1}{z};$$

$$\therefore\ \frac{dx}{xdz} = -\frac{2z}{1+z^2};$$

$$\therefore\ \log\left(\frac{x}{2c}\right) = \log\frac{1}{1+z^2} = \log\frac{x^2}{x^2+y^2};$$

$$\therefore\ x^2 + y^2 - 2cx = 0,$$

the equation to a circle, the origin being in the circumference. This is the solution of the problem: Find the curve in which the perpendicular from the origin upon the tangent is equal to the abscissa.

Ex. 2. $y - 2px + 4yp^2 = 0$; $\therefore\ y^2 = cx - c^2$.

115. *Integration of the equation, called Clairaut's Formula.*

$$y = px + f(p) = px + P.$$

Differentiate, when we have

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + \frac{dP}{dx};$$

$$\therefore \text{ since } \frac{dy}{dx} = p, \text{ and } \frac{dP}{dx} = P'\frac{dp}{dx}, \text{ we have}$$

$$0 = (x + P')\frac{dp}{dx}; \ \therefore\ \frac{dp}{dx} = 0, \text{ or } x + P' = 0.$$

$$\text{If we make } \frac{dp}{dx} = 0;\ p = c;\ \therefore\ y = cx + c'.$$

This equation appears to have two arbitrary constants; but if c be put for p in the original equation, and C for P, C being what P becomes when c is substituted for p, then $y = cx + C$; $\therefore\ C = c'$, and the equation has but one arbitrary constant. This is the general solution.

Again, from $x + P' = 0$, a value of p will be obtained which is a function of x or y, and does not introduce into

the original equation the constant by the elimination of which the differential equation was formed; such a solution of the equation is called a singular or particular solution.

The particular value may be derived from the general solution, by making c to vary; and as $y = cx + C$ is the equation to a straight line, the particular solution gives the equation to the curve which is the locus of the intersections of the straight lines denoted by the general solution.

Ex. 1. $y - px = a\sqrt{1+p^2}$;

$$\therefore \frac{dy}{dx} - p - x\frac{dp}{dx} = \frac{ap}{\sqrt{1+p^2}}\frac{dp}{dx};$$

$$\therefore \left\{x + \frac{ap}{\sqrt{1+p^2}}\right\} = 0, \text{ and } \frac{dp}{dx} = 0;$$

$$\therefore p = c, \text{ and } y = cx + a\sqrt{1+c^2},$$

which is the general solution.

$$\text{But } x = \frac{-ap}{\sqrt{1+p^2}}; \quad \therefore \frac{a^2}{x^2} = \frac{1+p^2}{p^2}; \quad \therefore \frac{\sqrt{a^2-x^2}}{x} = \frac{1}{p};$$

$$\therefore p = \frac{x}{\sqrt{a^2-x^2}}; \quad \sqrt{1+p^2} = \frac{-ap}{x} = \frac{-a}{\sqrt{a^2-x^2}};$$

$$\therefore y = \frac{x^2}{\sqrt{a^2-x^2}} - \frac{a^2}{\sqrt{a^2-x^2}} = -\frac{a^2-x^2}{\sqrt{a^2-x^2}}$$

$$= -\sqrt{a^2-x^2}; \quad \therefore y^2 + x^2 = a^2,$$

which is the solution of the problem: "Find the curve in which each of the perpendiculars drawn from a given point upon the tangent is equal to a given line."

Ex. 2. Let $y = px + \frac{a}{p}(1+p^2)$; $\therefore y^2 = 4a(a+x)$.

Ex. 3. $y = px + a\sqrt[3]{1+p^3}$; $y = cx + a\sqrt[3]{1+c^3}$.

Ex. 4. $y = px + \dfrac{ap}{\sqrt{1+p^2}}$; $\therefore y^{\frac{2}{3}} + x^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Ex. 5. Let $(y-px)^m = a^{m-1}\left(\dfrac{y}{p} - x\right)$;

$$\therefore \left(\frac{y}{m}\right)^m = \left(\frac{a}{m-1}\right)^{m-1} x;$$

this is the curve in which $AD^m = a^{m-1}AT$.

$$\therefore\ -\frac{dy_1}{y_1^2}+\frac{b}{y_1^2}dx=ax^m dx;\quad \therefore\ dy_1+ay_1^2x^m dx=bdx.$$

Let $x^{m+1}=x_1$; $\therefore\ x=x_1^{\frac{1}{m+1}}$; $\therefore\ dx=\dfrac{1}{m+1}x_1^{-\frac{m}{m+1}}dx_1$.

$$\text{And } dy_1+\frac{a}{m+1}y_1^2dx_1=\frac{b}{m+1}x_1^{-\frac{m}{m+1}}dx_1;$$

or putting $\dfrac{a}{m+1}=b_1$, $\dfrac{b}{m+1}=a_1$, and $-\dfrac{m}{m+1}=m_1$;

$$dy_1+b_1y_1^2dx_1=a_1x_1^{m_1}dx_1,$$

which may be integrated by the former method if $m_1 = \dfrac{-4n}{2n-1}$; or if $\dfrac{m}{m+1}=\dfrac{4n}{2n-1}$, whence $m=\dfrac{-4n}{2n+1}$. Hence Riccati's equation is integrable when m is of the form $\dfrac{-4n}{2n\mp 1}$. The first case belongs to the upper, the second to the lower sign.

Ex. 1. Integrate $dy+y^2dx=\dfrac{a^2dx}{x^{\frac{4}{3}}}$.

$$\text{Here } -\tfrac{4}{3} \text{ is of the form } -\frac{4n}{2n+1};$$

$$\therefore\ \text{let } y=\frac{1}{y_1},\ \text{and let } x^{m+1}=x^{-\frac{4}{3}+1}=x^{-\frac{1}{3}}=x_1;$$

$$\therefore\ x=x_1^{-3};\ \ dx=-3x_1^{-4}dx_1;\ \ x^{\frac{4}{3}}=x_1^{-4};$$

$$\therefore\ -\frac{dy_1}{y_1^2}-\frac{3}{y_1^2}x_1^{-4}dx_1=-3a^2dx_1,$$

$$dy_1-3a^2y_1^2dx_1=-3x_1^{-4}dx_1.$$

Let $-3a^2=b_1$, $-3=a_1$; $\therefore\ dy_1+b_1y_1^2dx_1=a_1x_1^{-4}dx_1$.

Now let $y_1=\dfrac{1}{b_1x_1}+\dfrac{z_1}{x_1^2}$.

Then $\dfrac{dz_1}{dx_1}+b_1z_1^2\dfrac{1}{x_1^2}=a_1x_1^{-2}$, or $x_1^2\dfrac{dz_1}{dx_1}=(a_1-b_1z^2)$;

$$\therefore\ \frac{1}{x_1^2}\frac{dx_1}{dz_1}=\frac{1}{a_1-b_1z_1^2}=\frac{1}{3(a^2z_1^2-1)},$$

$$\frac{3a}{x_1}=\log\sqrt{\frac{az_1+1}{az_1-1}}\times\frac{1}{c}=\log\frac{1}{c}\sqrt{\frac{3a^2x_1^2y_1+x_1+3a}{3a^2x_1^2y_1+x_1-3a}};$$

$$\therefore\ \text{since } \frac{1}{x_1}=x^{\frac{1}{3}} \text{ and } y_1=\frac{1}{y};$$

$$\therefore\ e^{6ax^{\frac{1}{3}}} = \frac{1}{c^2}\left\{\frac{3a^2x^{-\frac{2}{3}} + y\,(3a + x^{-\frac{1}{3}})}{3a^2x^{-\frac{2}{3}} + y\,(x^{-\frac{1}{3}} - 3a)}\right\}$$

$$= \frac{1}{c^2}\left\{\frac{3a^2x^{-\frac{1}{3}} + y\,(1 + 3ax^{\frac{1}{3}})}{3a^2x^{-\frac{1}{3}} + y\,(1 - 3ax^{\frac{1}{3}})}\right\};$$

$$\therefore\ c^2 = C = e^{-6ax^{\frac{1}{3}}}\left\{\frac{3a^2x^{-\frac{1}{3}} + y\,(1 + 3ax^{\frac{1}{3}})}{3a^2x^{-\frac{1}{3}} + y\,(1 - 3ax^{\frac{1}{3}})}\right\}.$$

Ex. 2. Let $dy + y^2dx = \dfrac{a^2dx}{x^{\frac{8}{3}}}$.

Here $-\dfrac{8}{3}$ is of the form $\dfrac{-4n}{2n-1}$;

$$\therefore\ \text{let } y = \frac{1}{x} + \frac{z}{x^2};\quad \text{and } \because b = 1,\ m = -\frac{8}{3};$$

$$\therefore\ \frac{dz}{dx} + \frac{bz^2}{x^2} = ax^{m+2} \text{ becomes } \frac{dz}{dx} + z^2\frac{1}{x^2} = a^2x^{-\frac{2}{3}}.$$

$$\text{Let } z = \frac{1}{y_1},\quad \text{and } \because m + 2 = -\frac{2}{3};\quad \therefore\ m + 1 = \frac{1}{3}.$$

$$\text{Let } x^{\frac{1}{3}} = x_1;\quad \therefore\ x^{-\frac{2}{3}}dx = 3dx_1;\quad \frac{1}{x} = \frac{1}{x_1^{\,3}};\quad \therefore\ \frac{dx}{x^2} = \frac{3dx_1}{x_1^{\,4}};$$

$$\therefore\ \frac{-dy_1}{y_1^{\,2}} + \frac{1}{y_1^{\,2}}\,\frac{3dx_1}{x_1^{\,4}} = 3a^2dx_1;\quad \therefore\ dy_1 + 3a^2y_1^{\,2}dx_1 = 3x_1^{\,-4}dx_1,$$

which, as has been shewn, is integrable.

110. It sometimes happens that equations in which the variables are separated admit of algebraical integrals, although the integral of each part is transcendental.

$$\text{Thus, since } \frac{dy}{\sqrt{1-y^2}} + \frac{dx}{\sqrt{1-x^2}} = 0,\quad (1);$$

$$\therefore\ \sin^{-1}x + \sin^{-1}y = C = \sin^{-1}c.$$

$$\therefore\ x\sqrt{1-y^2} + y\sqrt{1-x^2} = c;$$

we may however obtain the same result, thus

$$(1)\quad \times xy;\ \therefore\ \int\frac{xydx}{\sqrt{1-x^2}} + \int\frac{xydy}{\sqrt{1-y^2}} = 0;$$

or $-y\sqrt{1-x^2} + \int dy\sqrt{1-x^2} - x\sqrt{1-y^2} + \int dx\sqrt{1-x^2} = -c$;

which since $dy\sqrt{1-x^2} + dx\sqrt{1-y^2} = 0$, reduces itself to

$$y\sqrt{1-x^2} + x\sqrt{1-y^2} = c,$$

the required algebraic relation between y and x.

111. Again, let $\dfrac{dy}{\sqrt{a+by+cy^2}}+\dfrac{dx}{\sqrt{a+bx+cx^2}}=0$;

make y and x functions of another variable t, so that

$$\frac{dy}{dt}=\sqrt{a+by+cy^2};\quad \therefore \frac{dx}{dt}=-\sqrt{a+bx+cx^2};$$

ct $x+y=p$, then squaring and differentiating,

$$2\frac{d^2y}{dt^2}=b+2cy;\quad 2\frac{d^2x}{dt^2}=b+2cx;$$

$$\therefore 2\frac{d^2y}{dt^2}+2\frac{d^2x}{dt^2}=2\frac{d^2p}{dt^2}=2b+2cp.$$

Multiply both sides by $\dfrac{dp}{dt}$, and integrate;

$$\therefore \frac{dp^2}{dt^2}=C+2bp+cp^2;\ \therefore \frac{dp}{dt}=\frac{dx}{dt}+\frac{dy}{dt}=\sqrt{C+2bp+cp^2};$$

or $\sqrt{a+bx+cx^2}-\sqrt{a+by+cy^2}=\sqrt{C+2b(x+y)+c(x+y)^2}$.

Ex. 2. $\dfrac{dy}{\sqrt{a+by+cy^2+ey^3+fy^4}}+\dfrac{dx}{\sqrt{a+bx+cx^2+ex^3+fx^4}}=0.$

Let $\dfrac{dx}{dt}=\sqrt{a+bx+cx^2+ex^3+fx^4}=\sqrt{X}$;

$$\therefore \frac{dy}{dt}=-\sqrt{a+by+cy^2+ey^3+fy^4}=\sqrt{Y}.$$

Make $x+y=p$; $x-y=q$; whence squaring and differentiating,

$$\frac{d^2x}{dt^2}+\frac{d^2y}{dt^2}=\frac{d^2p}{dt^2}=\frac{1}{2}\left(\frac{dX}{dx}+\frac{dY}{dy}\right)$$

$$=b+c(x+y)+\tfrac{3}{2}e(x^2+y^2)+2f(x^3+y^3)$$

$$=b+cp+\tfrac{3}{4}e(p^2+q^2)+\tfrac{1}{2}fp(p^2+3q^2),$$

$$\frac{dx^2}{dt^2}-\frac{dy^2}{dt^2}=\frac{dp\,.\,dq}{dt^2}=bq+cpq+\frac{e}{4}(3p^2q+q^3)+\frac{f}{2}qp(p^2+q^2);$$

$$\therefore q\frac{d^2p}{dt^2}-\frac{dp\,.\,dq}{dt^2}=\frac{e}{2}q^3+fpq^3;$$

$\therefore$ multiplying both sides by $\dfrac{2}{q^3}\cdot\dfrac{dp}{dt}$;

$$\frac{2}{q^2}\cdot\frac{d^2p}{dt^2}\cdot\frac{dp}{dt}-\frac{1}{q^3}\cdot\frac{dp^2}{dt^2}\cdot 2\frac{dq}{dt}=e\frac{dp}{dt}+2fp\cdot\frac{dp}{dt};$$

whence integrating

$$\frac{1}{q^2}\frac{dp^2}{dt^2} = C + ep + fp^2; \quad \therefore \quad \frac{dp}{dt} = q\sqrt{C + ep + fp^2};$$

$$\text{or } \because \frac{dp}{dt} = \frac{dx}{dt} + \frac{dy}{dt}, \quad q = x - y, \quad \text{and } p = x + y,$$

$$\sqrt{a + bx + cx^2 + ex^3 + fx^4} - \sqrt{a + by + cy^2 + ey^3 + fy^4}$$
$$= (x - y)\sqrt{c + e(x + y) + f(x + y)^2},$$

an algebraic equation, which may be put also under a rational form: for writing $(x-y)\sqrt{P}$ instead of the right-hand side of the equation, inverting and multiplying by $X - Y$;

$$\therefore \frac{X - Y}{\sqrt{X} - \sqrt{Y}} = \frac{X - Y}{(x - y)\sqrt{P}};$$

$$\text{or } \sqrt{X} + \sqrt{Y} = \frac{X - Y}{(x - y)\sqrt{P}}; \quad \therefore \text{ by addition,}$$

$$2\sqrt{X} = (x - y)\sqrt{P} + \frac{X - Y}{(x - y)\sqrt{P}} = \frac{(x - y)^2 P + (X - Y)}{(x - y)\sqrt{P}};$$

and squaring both sides, the equation becomes rational.

Ex. 3. $\dfrac{d\phi}{\sqrt{1 - e^2\sin^2\phi}} + \dfrac{d\theta}{\sqrt{1 - e^2\sin^2\theta}} = 0.$

Let $\dfrac{d\phi}{dt} = \sqrt{1 - e^2\sin^2\phi}$; $\therefore$ $\dfrac{d\theta}{dt} = -\sqrt{1 - e^2\sin^2\theta}$;

$$\therefore \frac{d^2\phi}{dt^2} = -e^2\sin\phi \cdot \cos\phi; \quad \frac{d^2\theta}{dt^2} = -e^2\sin\theta \cdot \cos\theta;$$

$$\therefore \frac{d^2\phi}{dt^2} + \frac{d^2\theta}{dt^2} = -\frac{e^2}{2}(\sin 2\phi + \sin 2\theta),$$

$$\frac{d^2\phi}{dt^2} - \frac{d^2\theta}{dt^2} = -\frac{e^2}{2}(\sin 2\phi - \sin 2\theta);$$

$\therefore$ by making $p = \phi + \theta$, $q = \phi - \theta$,

$$\frac{d^2p}{dt^2} = -\frac{e^2}{2} \cdot \{\sin(p + q) + \sin(p - q)\} = -e^2\sin p \cdot \cos q,$$

$$\frac{d^2q}{dt^2} = -\frac{e^2}{2} \cdot \{\sin(p + q) - \sin(p - q)\} = -e^2 \cdot \sin q \cdot \cos p.$$

And $\dfrac{dp}{dt}\dfrac{dq}{dt} = \dfrac{d\phi^2}{dt^2} - \dfrac{d\theta^2}{dt^2} = -e^2(\sin^2\phi - \sin^2\theta)$

tween u and p, thence we obtain p in terms of u, and $\therefore$ x in terms of u from $\frac{dx}{x} = \frac{du}{p-u}$.

Ex. 1. $x^2 q = xp + 3y$; $y = ax^3 - \frac{b}{x}$.

Ex. 2. $x^3 q = (y - px)^2$; $y = x \log\left(\frac{a - cx}{x}\right)$.

119. To integrate the equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0.$$

Make $y = e^{\int_x u}$; $\therefore$ $\frac{dy}{dx} = ue^{\int_x u}$; $\frac{d^2y}{dx^2} = \left(\frac{du}{dx} + u^2\right)e^{\int_x u}$;

$$\therefore\ e^{\int_x u}\left\{u^2 + Pu + Q + \frac{du}{dx}\right\} = 0;$$

$$\text{whence } u^2 + Pu + Q + \frac{du}{dx} = 0,$$

an equation of the first degree and order; but which is seldom integrable when P and Q are functions of x. It however is, when P and Q are constant; let $P = A$; $Q = B$;

$$\therefore\ \frac{du}{dx} + u^2 + Au + B = 0;$$

$$\text{or } \frac{du}{dx} + (u - a)(u - b) = 0;$$

which is satisfied by making $u = a$ and $u = b$;

$$\therefore\ y = e^{\int_x a} = e^{ax + c'} = c_1 e^{ax},$$

$$\text{and } y = e^{\int_x b} = e^{bx + c''} = c_2 e^{bx};$$

either of these values substituted for y will satisfy the conditions of the differential equation; but the complete solution, which must comprise two constants, is $y = c_1 e^{ax} + c_2 e^{bx}$; which by substitution we find also satisfies it.

Cor. 1. If the roots of $u^2 + Au + B = 0$ be impossible,

$$a = \alpha + \beta\sqrt{-1}, \text{ and } b = \alpha - \beta\sqrt{-1};$$

$$\therefore\ y = e^{\alpha x}\{c_1 e^{\beta x\sqrt{-1}} + c_2 e^{-\beta x\sqrt{-1}}\}$$

$$= e^{\alpha x}\{(c_1 + c_2)\cos\beta x + (c_1 - c_2)\sqrt{-1}\sin\beta x\}.$$

Make $c_1 + c_2 = A\sin\delta$, $(c_1 - c_2)\sqrt{-1} = A\cos\delta$;

$$\therefore\ y = Ae^{\alpha x}\{\sin\delta\cos\beta x + \cos\delta\sin\beta x\} = Ae^{\alpha x}\sin(\beta x + \delta).$$

COR. 2. Let the roots be equal; or $a = b$.

Then $y = e^{ax}(c_1 + c_2) = c'e^{ax}$ which has but one constant. To find the second constant.

Suppose $b = a + h$; $\therefore\ y = c_1 e^{ax} + c_2 e^{ax+hx}$

$$= e^{ax}\{c_1 + c_2 e^{hx}\} = e^{ax}\left\{c_1 + c_2 + c_2 hx + \frac{c_2 h^2 x^2}{1 \cdot 2} + \&c.\right\};$$

$$\text{make } c_1 + c_2 = c', \quad c_2 h = c'', \quad \text{and } h = 0;$$

$$\therefore\ y = e^{ax}(c' + c''x).$$

120. The equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$, is seldom integrable when P and Q are functions of x; it can however be solved when $P = \frac{A}{a + bx}$; and $Q = \frac{B}{(a + bx)^2}$.

For make $a + bx = e^{bz}$;

$$\therefore\ \frac{dz}{dx} = \frac{1}{a + bx}; \quad \therefore\ \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\frac{1}{a + bx};$$

$$\therefore\ \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}\frac{dz}{dx}\frac{1}{a + bx} - \frac{dy}{dz}\frac{b}{(a + bx)^2}$$

$$= \left(\frac{d^2y}{dz^2} - b\frac{dy}{dz}\right)\frac{1}{(a + bx)^2};$$

whence by substitution, and multiplying by $(a + bx)^2$,

$$\frac{d^2y}{dz^2} + (A - b)\frac{dy}{dz} + By = 0;$$

which may be integrated by the preceding methods.

121. To integrate the general equation

$$\frac{d^ny}{dx^n} + A\frac{d^{n-1}y}{dx^{n-1}} + B\frac{d^{n-2}y}{dx^{n-2}} + \&c. + Ly = 0;$$

where A, B, C, &c. L, are constant.

Let $y = e^{mx}$; $\therefore\ \frac{dy}{dx} = me^{mx}$; $\frac{d^2y}{dx^2} = m^2e^{mx}$, &c.

$$\therefore\ m^n + Am^{n-1} + Bm^{n-2} + Cm^{n-3} + \&c. + L = 0.$$

Let a, b, c, &c. be the roots of this equation; then

$$y = e^{ax}, \quad y = e^{bx}, \quad y = e^{cx}; \ \&c.$$

will be particular integrals of the general equation, and the

substitution of each in it will satisfy it. Hence the complete integral will be, by the introduction of n constants,

$$y = c_1e^{ax} + c_2e^{bx} + c_3e^{cx} + \&c.$$

Cor. 1. Should any of the roots be equal, as $a = b$; then for $c_1e^{ax} + c_2e^{bx}$, put $e^{ax}(c_1 + c_2x)$;

$$\therefore\ y = e^{ax}(c_1 + c_2x) + c_3e^{cx} + \&c.$$

And if three roots be equal, and a be the equal root, put

$$e^{ax}(c_1 + c_2x + c_3x^2) \text{ for } c_1e^{ax} + c_2e^{bx} + c_3e^{cx},$$

and so on for any number of equal roots.

Cor. 2. If pairs of roots be impossible, substitute for the impossible exponential functions, the cosines and sines of the circular arcs, to which they are equivalent.

Ex. 1. $\dfrac{d^2u}{d\theta^2} + n^2u = 0.$

$$\text{Let } u = e^{m\theta};\quad \therefore\ \frac{du}{d\theta} = me^{m\theta};\quad \frac{d^2u}{d\theta^2} = m^2e^{m\theta};$$

$$\therefore\ m^2e^{m\theta} + n^2e^{m\theta} = 0;\quad \therefore\ m^2 + n^2 = 0, \text{ and } m = \pm n\sqrt{-1};$$

$$\begin{aligned}\therefore\ u &= c'e^{n\theta\sqrt{-1}} + c''e^{-n\theta\sqrt{-1}}\\ &= (c' + c'')\cos n\theta + (c' - c'')\sqrt{-1}\sin n\theta\\ &= A\cos(n\theta + B).\end{aligned}$$

If $c' + c'' = A\cos B$, and $(c' - c'')\sqrt{-1} = -A\sin B$.

Ex. 2. $\dfrac{d^2u}{d\theta^2} + n^2u + a^2 = 0.$

$$\text{Make } a^2 = n^2\beta, \text{ and } u + \beta = w;$$

$$\therefore\ \frac{d^2w}{d\theta^2} + n^2w = 0;\quad \therefore\ u = -\beta + A\cos(n\theta + B).$$

Ex. 3. $\dfrac{d^2s}{dt^2} + 2k\dfrac{ds}{dt} + fs = 0.$

$$\text{Make } s = e^{mt};\quad \therefore\ m^2 + 2km + f = 0;$$

$$\therefore\ m = -k \pm \sqrt{-1}\sqrt{f - k^2} = -k \pm a\sqrt{-1};$$

$$\therefore\ s = e^{-kt}(c'e^{at\sqrt{-1}} + c''e^{-at\sqrt{-1}}) = Ae^{-kt}\cos(at + B).$$

Examples (1) and (2) are useful in *Physical Astronomy;* Ex. (3) gives the space a function of the time, when a body moves through the arc of a cycloid, the resistance varying as the velocity.

Ex. 4. $\dfrac{d^3y}{dx^3} - 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} - 6y = 0.$

Let $y = e^{mx}$; $\therefore\ m^3 - 6m^2 + 11m - 6 = 0$;

$$\therefore\ y = c_1e^x + c_2e^{2x} + c_3e^{3x}.$$

Ex. 5. Let $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = 0.$

$$\therefore\ y = e^x(c_1 + c_2x + c_3x^2).$$

Ex. 6. $\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} + 16y = 0$; $\therefore\ y = e^{-4x}(c_1 + c_2x).$

Ex. 7. $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 34y = 0$; $\therefore\ y = Ae^{3x}\cos(B + 5x).$

Ex. 8. $\dfrac{d^2y}{dx^2} - \dfrac{1}{x}\dfrac{dy}{dx} + \dfrac{y}{x^2} = 0.$ Let $x = e^z$;

$$\therefore\ y = e^z(c_1 + c_2z) = x(c_1 + c_2\log x).$$

Ex. 9. $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} - \dfrac{y}{x^2} = 0$;

$$\therefore\ y = c_1e^z + c_2e^{-z} = c_1x + \frac{c_2}{x}.$$

Ex. 10. Integrate $\dfrac{d^4y}{dx^4} - a^4y = 0.$

$$\therefore\ y = c_1e^{ax} + c_2e^{-ax} + A\cos(B + ax).$$

Ex. 11. Integrate $\dfrac{d^4y}{dx^4} + a^4y = 0.$

Here $m^4 + a^4 = 0$; $\dfrac{m}{a} = \dfrac{1 \pm \sqrt{-1}}{\sqrt{2}}$, and $= \dfrac{-1 \pm \sqrt{-1}}{\sqrt{2}}$;

$$\therefore\ y = Ae^{\frac{ax}{\sqrt{2}}}\cos\left(B + \frac{ax}{\sqrt{2}}\right) + A_1e^{-\frac{ax}{\sqrt{2}}}\cos\left(B_1 + \frac{ax}{\sqrt{2}}\right).$$

Ex. 12. Integrate $\dfrac{d^4y}{dx^4} = \dfrac{1}{a^2}\dfrac{d^2y}{dx^2}.$

$$\therefore\ y = c_1e^{\frac{x}{a}} + c_2e^{-\frac{x}{a}} + c_3 + c_4x.$$

Ex. 13. Integrate $\dfrac{d^ny}{dx^n} - y = 0.$

Make $y = e^{mx}$; $\therefore\ m^n - 1 = 0,$

let 1, a_1, a_2, a_3, a_4, &c. a_{n-1}, be the roots of this equation

$$\therefore\ y = c_1 e^{x} + c_2 e^{a_1 x} + c_3 e^{a_2 x} + \&c. + c_n e^{a_{n-1} x}.$$

122. To solve the equation,

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \ \ldots\ldots\ldots\ldots (1).$$

We shall shew that the solution of this equation may be made to depend upon that of the equation,

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \ \ldots\ldots\ldots\ldots (2).$$

To effect this, we proceed to apply to this equation, a method called by Lagrange, "The Variation of the Parameters;" which consists in this, that if $y = c'y_1 + c''y_2$ be the solution of the equation (2), we may assume it to be that of equation (1), if c' and c'' be considered functions of x.

Let $\therefore\ y = c'y_1 + c''y_2$ be the solution of (1);

$$\therefore\ \frac{dy}{dx} = c'\frac{dy_1}{dx} + c''\frac{dy_2}{dx} + y_1\frac{dc'}{dx} + y_2\frac{dc''}{dx}.$$

But as we have made but one supposition to determine c' and c'', we may make another; let therefore

$$y_1\frac{dc'}{dx} + y_2\frac{dc''}{dx} = 0; \quad \therefore\ \frac{dy}{dx} = c'\frac{dy_1}{dx} + c''\frac{dy_2}{dx};$$

$$\therefore\ \frac{d^2y}{dx^2} = c'\frac{d^2y_1}{dx^2} + c''\frac{d^2y_2}{dx^2} + \frac{dc'}{dx}\frac{dy_1}{dx} + \frac{dc''}{dx}\frac{dy_2}{dx};$$

whence by substitution in the original equation (1),

$$c'\left(\frac{d^2y_1}{dx^2} + P\frac{dy_1}{dx} + Qy_1\right) + c''\left(\frac{d^2y_2}{dx^2} + P\frac{dy_2}{dx} + Qy_2\right)$$
$$+ \frac{dc'}{dx}\frac{dy_1}{dx} + \frac{dc''}{dx}\frac{dy_2}{dx} = R,$$

which by means of equation (2) is reduced to

$$\frac{dc'}{dx}\frac{dy_1}{dx} + \frac{dc''}{dx}\frac{dy_2}{dx} = R,$$

$$\text{or } \because\ \frac{dc''}{dx} = -\frac{y_1}{y_2}\frac{dc'}{dx},$$

$$\frac{dc'}{dx}\left(\frac{dy_1}{dx} - \frac{y_1}{y_2}\frac{dy_2}{dx}\right) = R;$$

whence $\frac{dc^1}{dx}$ is found to be a function of x, and $c' = X_1 + C_1$, also similarly $c'' = X_2 + C_2$;

$$\therefore\ y = C_1y_1 + C_2y_2 + y_1X_1 + y_2X_2.$$

A similar proof applies to equations of a higher order.

Ex. 1. Integrate $\frac{d^2y}{dx^2} + a^2y = \cos\beta x$.

The solution of the equation $\frac{d^2y}{dx^2} + a^2y = 0$ is

$$y = c'\cos ax + c''\sin ax;$$

let this be the solution of the proposed equation;

$$\therefore\ \frac{dy}{dx} = -c'a\sin ax + c''a\cos ax + \frac{dc'}{dx}\cos ax + \frac{dc''}{dx}\sin ax$$

$$= -c'a\sin ax + c''a\cos ax.$$

$$\text{Since } \frac{dc'}{dx}\cos ax + \frac{dc''}{dx}\sin ax = 0;$$

$$\therefore\ \frac{d^2y}{dx^2} = -c'a^2\cos ax - c''a^2\sin ax - a\frac{dc'}{dx}\sin ax + a\frac{dc''}{dx}\cos ax$$

$$= -a^2y - a\frac{dc'}{dx}\sin ax + a\frac{dc''}{dx}\cos ax;$$

$$\therefore\ -a\frac{dc'}{dx}\sin ax + a\frac{dc''}{dx}\cos ax = \cos\beta x\ldots\ldots(1).$$

$$\text{But } \frac{dc''}{dx} = -\frac{\cos ax}{\sin ax}\frac{dc'}{dx};$$

$$\therefore\ -a\frac{dc'}{dx}\left(\sin ax + \frac{\cos^2 ax}{\sin ax}\right) = \cos\beta x;$$

$$\therefore\ \frac{dc'}{dx} = -\frac{1}{a}\cos\beta x\sin ax = -\frac{1}{2a}\{\sin(a+\beta)x + \sin(a-\beta)x\},$$

$$\text{and } \frac{dc''}{dx} = \frac{1}{a}\cos\beta x\cos ax = \frac{1}{2a}\{\cos(a+\beta)x + \cos(a-\beta)x\};$$

$$\therefore\ c' = c_1 + \frac{1}{2a}\left\{\frac{\cos(a+\beta)x}{a+\beta} + \frac{\cos(a-\beta)x}{a-\beta}\right\},$$

$$c'' = c_2 + \frac{1}{2a}\left\{\frac{\sin(a+\beta)x}{a+\beta} + \frac{\sin(a-\beta)x}{a-\beta}\right\};$$

$$\therefore\ y = c_1\cos ax + c_2\sin ax + \frac{1}{2a}\left(\frac{\cos\beta x}{a+\beta} + \frac{\cos\beta x}{a-\beta}\right)$$

$$= c_1\cos ax + c_2\sin ax + \frac{\cos\beta x}{a^2-\beta^2}.$$

Ex. 2. Integrate $\dfrac{d^2y}{dx^2} + a^2y = X.$

Let $y = c' \cos ax + c'' \sin ax$, be the solution. Proceeding as in Example 1,

$$\therefore \frac{dc'}{dx}\sin ax - \frac{dc''}{dx}\cos ax = -\frac{X}{a}.$$

$$\text{And } \frac{dc''}{dx} = -\frac{dc'}{dx}\frac{\cos ax}{\sin ax};$$

$$\therefore \frac{dc'}{dx} = -\frac{1}{a}X\sin ax, \text{ and } \frac{dc''}{dx} = \frac{1}{a}X\cos ax;$$

$$\therefore c' = c_1 - \frac{1}{a}\int_x X\sin ax,$$

$$c'' = c_2 + \frac{1}{a}\int_x X\cos ax;$$

$$\therefore y = c_1\cos ax + c_2\sin ax - \frac{\cos ax}{a}\int_x X\sin ax$$

$$+\frac{\sin ax}{a}\int_x X\cos ax.$$

Ex. 3. Integrate $\dfrac{d^2y}{dx^2} + A\dfrac{dy}{dx} + By = X;\quad X = f(x).$

Let a and b be the roots of the equation $m^2 + Am + B = 0$;
$\therefore$ let $y = c'e^{ax} + c''e^{bx}$ be the solution of the equation;

$$\therefore \frac{dy}{dx} = ac'e^{ax} + bc''e^{bx} + e^{ax}\frac{dc'}{dx} + e^{bx}\frac{dc''}{dx}.$$

$$\text{Make } e^{ax}\frac{dc'}{dx} + e^{bx}\frac{dc''}{dx} = 0;$$

$$\therefore \frac{dy}{dx} = ac'e^{ax} + bc''c^{bx}.$$

$$\frac{d^2y}{dx^2} = a^2c'e^{ax} + b^2c''e^{bx} + ae^{ax}\frac{dc'}{dx} + be^{bx}\frac{dc''}{dx};$$

$$\therefore c'e^{ax}(a^2 + Aa + B) + c''e^{bx}(b^2 + Ab + B)$$

$$+ae^{ax}\frac{dc'}{dx} + be^{bx}\frac{dc''}{dx} = X.$$

And $a^2 + Aa + B = 0;\quad b^2 + Ab + B = 0;$

$$\therefore ae^{ax}\frac{dc'}{dx} + be^{bx}\frac{dc''}{dx} = X.$$

But $be^{ax}\frac{dc'}{dx} + be^{bx}\frac{dc''}{dx} = 0$;

$$\therefore\ (a-b)e^{ax}\frac{dc'}{dx} = X, \text{ and } -(a-b)e^{bx}\frac{dc''}{dx} = X;$$

$$\therefore\ c' = c_1 + \frac{1}{a-b}\int_x Xe^{-ax},\ c'' = c_2 - \frac{1}{a-b}\int_x Xe^{-bx};$$

$$\therefore\ y = c_1e^{ax} + c_2e^{bx} + \frac{e^{ax}}{a-b}\int_x Xe^{-ax} - \frac{e^{bx}}{a-b}\int_x Xe^{-bx}.$$

Ex. 4. Integrate $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x$.

$$\therefore\ y = c_1e^{3x} + c_2e^{2x} + \frac{1}{6}\left(x + \frac{5}{6}\right).$$

Simultaneous Differential Equations.

123. In the applications of the Differential Calculus to physical problems, mutually dependent equations are frequently found in which $n+1$ variables are involved, and n equations are given: as most commonly the unknown quantities are x, y, and t; x and y being functions of t; we shall first solve the system of equations which involve these quantities. The method of solution is due to D'Alembert.

$$\text{Let } A\frac{dx}{dt} + B\frac{dy}{dt} + Cx + Dy = \theta,$$

$$\text{and } A_1\frac{dx}{dt} + B_1\frac{dy}{dt} + C_1x + D_1y = \theta_1,$$

be the two equations: A, B, &c. being constant, θ and θ_1 functions of t. By the successive elimination of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ these may be reduced to the form,

$$\frac{dx}{dt} + ax + by = T\ (1),\quad \frac{dy}{dt} + a_1x + b_1y = T_1\ (2).$$

Now multiply (2) by m and add the product to (1);

$$\therefore\ \frac{d}{dt}(x+my) + (a+ma_1)\left(x + \frac{b+mb_1}{a+ma_1}y\right) = T + mT_1.$$

Let $\frac{b+mb_1}{a+ma_1} = m$: and let m_1, m_2 be the two values of m

resulting from the equation; also let $a+m_1a_1=r_1$; $a+m_2a_1=r_2$;
$\therefore$ we shall have the two linear equations of the first order,

$$\frac{d}{dt}(x+m_1y)+r_1(x+m_1y)=T+m_1T_1;$$

$$\frac{d}{dt}(x+m_2y)+r_2(x+m_2y)=T+m_2T_1;$$

$$\therefore\ x+m_1y=e^{-r_1t}\{\textstyle\int_t e^{r_1t}(T+m_1T_1)+C\},$$

$$x+m_2y=e^{-r_2t}\{\textstyle\int_t e^{r_2t}(T+m_2T_1)+C_1\}.$$

Ex. Let $\dfrac{dx}{dt}+4y+5x=e^t$; $\dfrac{dy}{dt}+x+2y=e^{2t}$;

$$\therefore\ \frac{d}{dt}(x+my)+(4+2m)y+(5+m)x=e^t+me^{2t}.$$

Let $\dfrac{4+2m}{5+m}=m$; $\therefore\ m=1$ or -4, $5+m=6$ or 1;

$$\therefore\ x+y=e^{-6t}\{\textstyle\int_t e^{6t}(e^t+e^{2t})+C\}=\frac{1}{7}e^t+\frac{1}{8}e^{2t}+Ce^{-6t},$$

$$\text{or } x-4y=e^{-t}\{\textstyle\int_t e^t(e^t+e^{2t})+C_1\}=\frac{1}{2}e^t+\frac{1}{3}e^{2t}+C_1e^{-t};$$

which give x and y in terms of t.

124. Next to integrate the simultaneous equations,

$$\frac{dx}{dt}+(Ax+By+Cz)=T\quad(1),$$

$$\frac{dy}{dt}+(A_1x+B_1y+C_1z)=T_1\quad(2),$$

$$\frac{dz}{dt}+(A_2x+B_2y+C_2z)=T_2\quad(3):$$

where A, B, C, &c. are constant, and T, T_1, T_2 functions of t: multiply (2) by m and (3) by m' and add;

$$\therefore\ \frac{d}{dt}(x+my+m'z)+(A+A_1m+A_2m')\{x+B'y+C'z\}=U,$$

$$\text{where } B'=\frac{B+mB_1+m'B_2}{A+A_1m+A_2m'},\quad C'=\frac{C+mC_1+m'C_2}{A+mA_1+m'A_2},$$

$$U=T+T_1m+T_2m';$$

$\therefore$ if $B'=m$; $C'=m'$; $A+A_1m+A_2m'=M$; $x+my+m'z=v$;

$$\frac{dv}{dt}+Mv=U, \text{ a linear equation,}$$

which integrated will give the relation between v and t: also since from $B'=m$, and $C'=m$, two cubic equations will

arise if m_1, m_2, m_3; m_1', m_2', m_3' be their roots, and if U_1, U_2, U_3 be the values of the right-hand side of the equation when integrated;

$$\therefore\ x + m_1y + m_1'z = U_1,$$
$$x + m_2y + m_2'z = U_2,$$
$$x + m_3y + m_3'z = U_3.$$

125. To integrate the simultaneous equation of the second order,

$$\frac{d^2x}{dt^2} + ax + by + c = 0 \quad (1),$$
$$\frac{d^2y}{dt^2} + a_1x + b_1y + c_1 = 0 \quad (2);$$

multiply (2) by m and add,

$$\therefore\ \frac{d^2}{dt^2}(x + my + c) + (a + ma_1)\left\{x + \frac{b + mb_1}{a + ma_1}y + \frac{c + mc_1}{a + ma_1}\right\} = 0.$$

Make $m = \dfrac{b + b_1m}{a + ma_1}$; $u = x + my + c$; $a + a_1m = -n^2$;

$$\therefore\ \frac{d^2u}{dt^2} - n^2u = 0; \quad \therefore\ u = Ce^{nt} + C_1e^{-nt};$$

therefore if m_1 and m_2 be the two values of m,

$$(a + a_1m_1)(x + m_1y) + c + m_1c_1 = (a + m_1a_1)\{c_1e^{nt} + c_2e^{-nt}\},$$
$$(a + a_1m_2)(x + m_2y) + c + m_2c_1 = (a + m_2a_1)\{c_1'e^{nt} + c_2'e^{-nt}\}.$$

Ex. $\dfrac{d^2x}{dt^2} = 3x + 4y - 3$: $\dfrac{d^2y}{dt^2} = 8y - x - 5$;

$$\therefore\ x = \frac{1}{7} + 4c_1e^{3t} + 4c_2e^{-3t} - 3c_1'e^{t\sqrt{7}} - c_2'e^{t\sqrt{7}},$$
$$y = \frac{9}{14} + c_1e^{3t} + c_2e^{-3t} - c_1'e^{t\sqrt{7}} - c_2'e^{-t\sqrt{7}}.$$

Differential Equations containing more than two Variables.

126. Equations of this description, of the first order, which, we in general shall suppose, involve the three variables x, y, z, may be divided into (1) Total Differential Equations, (2) Partial Differential Equations.

Total Differential Equations.

Let $du = Pdx + Qdy + Rdz$ be the equation, which may be supposed to arise from the differentiation of

$$u = f(xyz)\,; \text{ whence } P = \frac{du}{dx}; \ Q = \frac{du}{dy}; \ R = \frac{du}{dz}:$$

and since when this is the case,

$$\frac{dP}{dy} = \frac{dQ}{dx}; \ \frac{dP}{dz} = \frac{dR}{dx}; \text{ and } \frac{dQ}{dz} = \frac{dR}{dy}:$$

we can always ascertain when an equation is a total differential.

Ex. Let $du = \dfrac{y}{a-z}\,dx + \dfrac{x}{a-z}\,dy + \dfrac{xy}{(a-z)^2}\,dz.$

$$\text{Here } \frac{dP}{dy} = \frac{1}{a-z} = \frac{dQ}{dx}, \ \frac{dP}{dz} = \frac{y}{(a-z)^2} = \frac{dR}{dx};$$

$$\therefore \frac{du}{dx} = \frac{y}{a-z}; \quad \therefore u = \frac{xy}{a-z} + f(yz);$$

$$\therefore \frac{du}{dy}.dy + \frac{du}{dz}dz$$

$$= \frac{x}{a-z}dy + \frac{xy}{(a-z)^2}dz + \frac{d}{dy}f(yz)\,dy + \frac{d}{dz}f(yz)\,dz;$$

$$\therefore \frac{d}{dy}f(yz)\,dy + \frac{d}{dz}f(yz)\,dz = 0;$$

$$\therefore f(yz) = C;$$

$$\therefore u = \frac{xy}{a-z} + C.$$

127. Next to integrate the equation $Pdx + Qdy + Rdz = 0$, which may be put under the form $dz = -\dfrac{P}{R}dx - \dfrac{Q}{R}dy$;

or which, by making $p = -\dfrac{P}{R}$, $q = -\dfrac{Q}{R}$,

may be written $dz = pdx + qdy$.

Now if this equation can be expressed by an equation

$$z = f(x, y, c), \text{ or } f(x, y, z) = c,$$

we ought to have $\dfrac{d(p)}{dy} = \dfrac{d(q)}{dx}$,

$$\text{or } \frac{dp}{dy} + \frac{dp}{dz} \cdot \frac{dz}{dy} = \frac{dq}{dx} + \frac{dq}{dz} \cdot \frac{dz}{dx},$$

$$\text{or } \because \frac{dz}{dy} = q, \text{ and } \frac{dz}{dx} = p, \text{ we have}$$

$$\frac{dp}{dy} - \frac{dq}{dx} + q\frac{dp}{dz} - p\frac{dq}{dz} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

an equation of condition, by which we can ascertain whether the proposed equation admits of the solution $f(x, y, z) = c$. If we restore the values of p and q the equation becomes

$$P\left(\frac{dQ}{dz} - \frac{dR}{dy}\right) + Q.\left(\frac{dR}{dx} - \frac{dP}{dz}\right) + R.\left(\frac{dP}{dy} - \frac{dQ}{dx}\right) = 0 \ldots (2);$$

when this equation or the preceding one holds, one of the variables must be considered as constant, and the remaining part of the equation integrated according to the rules given for the integration of functions of two variables.

Ex. 1. $(y + z)\,dx + (x + y)\,dz + (z + x)\,dy = 0.$

$$\text{Here } P = y + z, \ \frac{dP}{dy} = 1 = \frac{dP}{dz}, \ \ Q = x + z, \ \frac{dQ}{dx} = \frac{dQ}{dz} = 1;$$

$$R = x + y; \ \frac{dR}{dx} = 1 = \frac{dR}{dy}; \ \therefore \text{ equation (2) is satisfied};$$

$$\therefore \text{ making } dz = 0; \ \ \frac{dx}{x + z} + \frac{dy}{y + z} = 0;$$

$$\therefore \log(x + z) + \log(y + z) = \phi(z) = \log(Z);$$

$$\therefore (x + z).(y + z) = Z;$$

$$\therefore (y + z)\,dx + (x + z)\,dy + (x + y + 2z)\,dz = dZ;$$

$$\therefore \frac{dZ}{dz} = 2z; \ \therefore Z = z^2 + C; \ \ xy + xz + yz = C.$$

Ex. 2. $(ay - bz)\,dx + (cz - ax)\,dy + (bx - cy)\,dz = 0.$
Make z constant; $\therefore dz = 0$, then

$$\frac{adx}{cz - ax} + \frac{ady}{ay - bz} = 0; \ \ \therefore \log\left(\frac{ay - bz}{cz - ax}\right) = \log Z;$$

$$\therefore \frac{ady}{cz - ax} + \frac{a(ay - bz)\,dx}{(cz - ax)^2} + \frac{a(bx - cy)}{(cz - ax)^2}\,dz = dZ;$$

$$\therefore (cz - ax)\,dy + (ay - bz)\,dx + (bx - cy)\,dz = 0$$

$$= \{(cz - ax)\}^2 dZ;$$

$$\therefore dZ = 0; \ \ \therefore Z = C'; \ \ \therefore (ay - bz) = C'(cz - ax).$$

128. If the equation $Pdx + Qdy + Rdz = 0$ is not a complete differential, but may be rendered so by the means of a factor F, the equation (2) must still be satisfied: for, multiplying by F,

$FPdx + FQdy + FRdz$ is an exact differential;

$$\therefore \frac{d.FP}{dy} = \frac{d.FQ}{dx}; \quad \frac{d.FR}{dx} = \frac{d.FP}{dz}; \quad \frac{d.FQ}{dz} = \frac{d.FR}{dy};$$

$$\therefore F\left(\frac{dP}{dy} - \frac{dQ}{dx}\right) + P.\frac{dF}{dy} - Q.\frac{dF}{dx} = 0,$$

$$F\left(\frac{dR}{dx} - \frac{dP}{dz}\right) + R.\frac{dF}{dx} - P.\frac{dF}{dz} = 0,$$

$$F\left(\frac{dQ}{dz} - \frac{dR}{dy}\right) + Q.\frac{dF}{dz} - R.\frac{dF}{dy} = 0.$$

Multiply the first of these equations by R, the second by Q, and the third by P, and add: we have

$$R\left(\frac{dP}{dy} - \frac{dQ}{dx}\right) + Q.\left(\frac{dR}{dx} - \frac{dP}{dz}\right) + P.\left(\frac{dQ}{dz} - \frac{dR}{dy}\right) = 0,$$

the same equation as in the preceding article.

When the differentials dx, dy, dz exceed the first degree, it must be solved with respect to dz; and can then only be integrated when the factors of the equation so solved are of the form $dz - pdx - qdy = 0$.

Partial Differential Equations.

129. It is here required to find $z = f(xy)$ from one of the partial differential coefficients, or from some relation existing between them.

To integrate $\frac{dz}{dx} = P$; P being a function of x, y, z; we first integrate it on the supposition that y is constant, and instead of adding an arbitrary constant after the integration we add $\phi(y)$: similarly we add $\phi(x)$, if the equation be $\frac{dz}{dy} = P$.

Ex. 1. $\frac{dz}{dx} = a; \quad \therefore z = ax + \phi(y).$

Ex. 2. $\frac{dz}{dx} = \frac{az}{x}; \quad \therefore \log z = \log x^a \phi(y); \quad \therefore z = x^a \phi(y).$

Ex. 3. $\frac{dz}{dx} = \frac{y^2 + z^2}{x^2 + y^2}$; $\therefore \frac{dz}{y^2 + z^2} - \frac{dx}{x^2 + y^2} = 0$;

$$\therefore \tan^{-1}\frac{z}{y} - \tan^{-1}\frac{x}{y} = \tan^{-1}\phi(y); \quad \frac{y(z-x)}{y^2 + xz} = \phi(y).$$

Ex. 4. $\frac{dz}{dx} = \frac{x^2 - y^2}{x^2 + y^2} \cdot \frac{z}{x}$; $z = \frac{x^2 + y^2}{x}\phi(y)$.

130. To integrate the equation $Pp + Qq = R$, in which P, Q, R, contain at once, x, y, z;

$$\because dz = pdx + qdy; \quad \therefore p = \frac{dz - qdy}{dx};$$

$$\therefore \text{substituting}, \quad Pdz - Rdx = q \cdot (Pdy - Qdx).$$

Here there are two cases: 1st, $Pdz - Rdx$ may contain only z and x, $Pdy - Qdx$ only y and x; 2nd, either or both of these factors may contain all the variables.

Case 1. Let F be the factor which will make $Pdz - Rdx$ a complete differential dM, and F_1 the factor which will make $Pdy - Qdx = dN$;

$\therefore dM = \frac{q \cdot F}{F_1} \cdot dN$, which cannot be integrated unless $\frac{qF}{F_1}$ is a function of N, $= \phi'(N)$; whence $dM = \phi'(N)dN$; $\therefore M = \phi(N)$.

Ex. 1. $px + qy = nz$; also $p = \frac{dz - qdy}{dx}$;

$$\therefore xdz - nzdx = q(xdy - ydx);$$

and to integrate $xdz - nzdx$, and $xdy - ydx$, we must multiply the former by $\frac{1}{x^{n+1}}$, the latter by $\frac{1}{x^2}$;

$$\therefore M = \frac{z}{x^n}; \quad N = \frac{y}{x}; \quad \therefore z = x^n\phi\left(\frac{y}{x}\right).$$

Ex. 2. $px + qy = 0$; $\therefore z = \phi\left(\frac{y}{x}\right)$.

Ex. 3. $qx - py = 0$; $\therefore z = \phi(x^2 + y^2)$.

Ex. 4. $ap + bq = c$; $\therefore z = \frac{cx}{a} + \frac{1}{a}\phi \cdot (ay - bx)$.

Ex. 5. $px - q = x^2$; $\therefore z = \frac{x^2}{2} + \phi \cdot (y + \log x)$.

CASE 2. Next let the variables x, y, z be found in both of the functions $Pdy-Qdx$, and $Pdz-Rdx$; we can no longer integrate them separately, since z cannot be considered constant in the former, nor x in the latter.

Lagrange observed that if these equations were integrated conjointly: and if we call the integral of the former N: and of the latter M; so that $N=a$, and $M=b$, a and b being arbitrary constants; then the complete integral will be $M=\phi(N)$. But that this method may succeed, one of the equations must involve two of the variables only; and its integral will enable us to eliminate one of the three variables x, y, z, from the remaining equation.

The truth of this proposition may be thus shewn.

Since the equations $N=a$, and $M=b$, are derived from $Pdx-Qdy=0$, and $Pdz-Rdx=0$, the differentials of $N=a$ and $M=b$ will be satisfied by the values of dz, dy, deduced from these latter equations: hence differentiating and putting

$$M_x=\frac{dM}{dx},\quad M_y=\frac{dM}{dy},\quad M_z=\frac{dM}{dz};$$

$$M_xdx+M_ydy+M_zdz=0;\ \text{ and }\ N_xdx+N_ydy+N_zdz=0.$$

But $\dfrac{dy}{dx}=\dfrac{Q}{P}$; $\dfrac{dz}{dx}=\dfrac{R}{P}$; $\therefore$ substituting

$$M_x=-M_y\cdot\frac{Q}{P}-M_z\cdot\frac{R}{P};\ N_x=-N_y\cdot\frac{Q}{P}-N_z\cdot\frac{R}{P}.$$

But from the equation $M=\phi(N)$;

$$\therefore\ M_xdx+M_ydy+M_zdz=\phi'(N)\{N_xdx+N_ydy+N_zdz\};$$

hence $M_y.(Pdy-Qdx)+M_z.(Pdz-Rdx)$

$$=\phi'(N)\{N_y(Pdy-Qdx)+N_z(Pdz-Rdx)\};$$

$$\therefore\ Pdz-Rdx=-\frac{M_y-N_y\phi'(N)}{M_z-N_z\phi'(N)}(Pdy-Qdx)=-\omega(Pdy-Qdx);$$

$$\therefore\ dz=\frac{R+\omega.Q}{P}dx-\omega.dy;$$

whence $p=\dfrac{R+\omega Q}{P}$, $q=-\omega$: which substituted in the original equation $Pp+Qq=R$ satisfy it; and therefore the assumption that $M=\phi(N)$ (which is derived from the integration of $Pdy-Qdx=0$; and $Pdz-Rdx=0$) is the solution of the problem, is completely justified.

Ex. 1. $px^2 - qxy + y^2 = 0$;

$\therefore\ x^2dz + y^2dx = q(x^2dy + yxdx)$;

$\therefore\ x^2dy + yxdx = 0\ (1)$; and $x^2dz + y^2dx = 0\ (2)$:

from (1), $xdy + ydx = 0$; $\therefore\ xy = a = N$; $\therefore\ y = \dfrac{a}{x}$:

from (2), $x^2dz + \dfrac{a^2}{x^2}dx = 0$; $\therefore\ z - \dfrac{a^2}{3x^3} = b = M$,

or $z - \dfrac{y^2}{3x} = \phi(N)$; $\therefore\ z = \dfrac{y^2}{3x} + \phi(xy)$.

Ex. 2. $px + qy = n\sqrt{x^2 + y^2}$;

$\therefore\ xdz - n\sqrt{x^2 + y^2}dx = q\,.\,(xdy - ydx)$:

from $xdy - ydx = 0$; $\dfrac{y}{x} = a = N$; $\therefore\ y = ax$;

$\therefore\ xdz - n\sqrt{x^2 + a^2x^2}dx = 0$; $\therefore\ z - nx\sqrt{1 + a^2} = b = M$;

$\therefore\ z - n\sqrt{x^2 + y^2} = \phi(N)$; $z = n\sqrt{x^2 + y^2} + \phi\left(\dfrac{y}{x}\right)$.

Ex. 3. $px^2 + qy - y^2 = 0$; $z = \dfrac{y^2}{2} + \phi\left(\dfrac{1}{x} + \log y\right)$.

Ex. 4. $qx + py = nz$; $z = (x + y)^n \,.\, \phi(y^2 - x^2)$.

This equation as well as the more general one $Pp + Qq = Rz$, is best solved by making $z = e^u$.

Ex. 5. $px^2 + qy^2 = nxy$; $z = \dfrac{nxy}{x - y}\log\left(\dfrac{x}{y}\right) + \phi\left(\dfrac{y - x}{xy}\right)$.

Ex. 6. $p + q\,.\,\dfrac{y}{x} = x^m z^n$; $\dfrac{1}{(n-1)z^{n-1}} + \dfrac{x^{m+1}}{m+1} = \phi\left(\dfrac{y}{x}\right)$.

Ex. 7. $px + qz + y = 0$; $x e^{\sin^{-1}\cdot\frac{z}{\sqrt{y^2+z^2}}} = \phi(y^2 + z^2)$.

Ex. 8. $(c - z) = (a - x)p + (b - y)q$; $\therefore\ \dfrac{y - b}{z - c} = \phi\left(\dfrac{x - a}{z - c}\right)$.

Ex. 9. $px + qy = \dfrac{xy}{z}$; $z^2 = xy + f\left(\dfrac{y}{x}\right)$.

Ex. 10. $z - px - qy = m(x + y + z)$;

$\therefore\ x^{m-1}(x + y + mz) = c^m\phi\left(\dfrac{y}{x}\right)$.

Ex. 11. $(y-bz)p-(x-az)q=bx-ay$;

$$\therefore (y-bz)dy+(x-az)dx=0 \ldots\ldots (1),$$
$$(y-bz)dz-(bx-ay)dx=0 \ldots\ldots (2),$$
$$(x-az)dz+(bx-ay)dy=0 \ldots\ldots (3);$$
$$\therefore (2)\times a-(3)\times b \text{ and } \div \text{ by } bx-ay;$$
$$\therefore dz+adx+bdy=0.$$
$$(2)\times x-3\times y \text{ and } \div \text{ by } bx-ay;$$
$$\therefore xdx+ydy+zdz=0;$$
$$\therefore x^2+y^2+z^2=\phi(z+ax+by),$$

this is the general equation to surfaces of revolution.

131. The same method applies to partial differential equations containing a greater number of variables.

Ex. 1. Let $nu+px+qy=az$,

$$\text{where } n=\frac{dz}{du} \text{ and } z=f(xyu):$$
$$\because dz=pdx+qdy+ndu; \quad \therefore p=\frac{dz-qdy-ndu}{dx};$$
$$\therefore n(udx-xdu)+q(ydx-xdy)+xdz-azdx=0;$$
$$\therefore udx-xdu=0; \quad \therefore \frac{x}{u}=\alpha,$$
$$ydx-xdy=0; \quad \therefore \frac{x}{y}=\beta,$$
$$xdz-azdx=0; \quad \therefore \frac{z}{x^a}=\gamma;$$
$$\therefore \text{since } \gamma=\phi(\alpha\beta); \quad z=x^a.\phi\left(\frac{x}{u},\ \frac{x}{y}\right).$$

Ex. 2. $ap+bq+cn=0$; put $n=\dfrac{dz-pdx-qdy}{du}$;

$$\therefore z=\phi\{(cx-au),\ (cy-bu)\}.$$

132. When the partial differential coefficients p and q exceed the first degree, q must be considered as a function of p, x, y, z, and the values of $\frac{dq}{dx}$, $\frac{dq}{dz}$ substituted in the equation $\frac{dp}{dy}-\frac{dq}{dx}+q\frac{dp}{dz}-p\frac{dq}{dz}=0$, which is derived from $dz=pdx+qdy$, considered to be a complete differential.

This substitution will give an equation which integrated will give a value of p, and $\therefore$ of q, in terms of x, y, z, and a constant a: substitute for p and q in $dz = pdx + qdy$, and integrate, whence $f(x, y, z, a) = \beta = \phi(a)$; and to eliminate (a) differentiate $b = \phi(a)$ with regard to (a).

Ex. 1. Let $\frac{dz}{dx} \cdot \frac{dz}{dy} = 1$, or $pq = 1$;

$$\therefore q = \frac{1}{p}; \quad \therefore \frac{dq}{dz} = -\frac{1}{p^2} \cdot \frac{dp}{dz}; \quad \frac{dq}{dx} = -\frac{1}{p^2} \cdot \frac{dp}{dx};$$

$$\therefore \text{ substituting in } \frac{dp}{dy} - \frac{dq}{dx} + q\frac{dp}{dz} - p\frac{dq}{dz} = 0,$$

$$\frac{dp}{dy} + \frac{1}{p^2}\frac{dp}{dx} + \frac{2}{p} \cdot \frac{dp}{dz} = 0;$$

$$\text{or if } \frac{dp}{dy} = q'; \quad \frac{dp}{dx} = p'; \quad \frac{dp}{dz} = n';$$

$$q' + \frac{1}{p^2}p' + \frac{2}{p}n' = 0;$$

whence, treating it as an equation of three variables and function of $(pxyz)$,

$$dp = 0; \quad \therefore p = a; \quad q = \frac{1}{a};$$

$$\therefore dz - adx - \frac{dy}{a} = 0; \quad \therefore z - ax - \frac{y}{a} = (\phi a);$$

whence, by differentiation, $\frac{y}{a^2} - x = \phi'(a)$.

Ex. 2. Let $\frac{dz^2}{dx^2} + \frac{dz^2}{dy^2} = 1$; or $p^2 + q^2 = 1$;

$$\therefore z - ax - y\sqrt{1 - a^2} = \phi(a),$$

$$x - \frac{ay}{\sqrt{1 - a^2}} = -\phi'(a).$$

Partial Differential Equations of the second Order.

133. Here make $\frac{d^2z}{dx^2} = r$; $\frac{d^2z}{dxdy} = s$; $\frac{d^2z}{dy^2} = t$.

To integrate $\frac{d^2z}{dx^2} = M$, M being a function of x and y:

$$\because p=\frac{dz}{dx}; \quad \therefore \frac{dp}{dx}=\frac{d^2z}{dx^2}=M; \quad \therefore p=\int_x M+\phi(y);$$
$$\therefore z=\int_x.\int_x M+x\phi(y)+f(y).$$

If $\dfrac{d^2z}{dxdy}=M$: then $\dfrac{dz}{dx}=\int_y M+\phi'(x)$;
$$\therefore z=\int_x\int_y M+\phi(x)+f(y).$$

Ex. 1. Let $\dfrac{d^2z}{dxdy}=xy$; $\therefore p=\dfrac{xy^2}{2}+\phi'(x)$;
$$\therefore z=\frac{x^2y^2}{4}+\phi(x)+f(y).$$

Ex. 2. $\dfrac{d^2z}{dxdy}=ax+by$; $\therefore z=\dfrac{ax^2y}{2}+\dfrac{by^2x}{2}+\phi(x)+f(y)$.

134. Let $\dfrac{d^2z}{dx^2}=P.\dfrac{dz}{dx}$, P being a function of x and y;
$$\therefore \frac{dp}{dx}=Pp; \quad \therefore \log p=\int_x P+\phi(y);$$
$$\therefore p=e^{\int_x P+\phi(y)}; \quad \therefore z=\int_x e^{\int_x P+\phi(y)}+f(y).$$

Ex. $x\dfrac{d^2z}{dx^2}=(n-1)\dfrac{dz}{dx}$; $nz=x^n\phi(y)+nf(y)$.

135. To integrate $\dfrac{d^2z}{dxdy}=P\dfrac{dz}{dx}$ or $\dfrac{dp}{dy}=Pp$;
$$\therefore z=\int_x e^{\int_y P+\phi(x)}+f(y).$$

136. To integrate $\dfrac{d^2z}{dx^2}=P\dfrac{dz}{dx}+Q$, P and Q being functions of x and y.

If $\dfrac{dz}{dx}=p$; $\therefore \dfrac{dp}{dx}-Pp=Q$, a linear equation;

whence $p=e^u\{\int_x e^{-u}Q+\phi(y)\}$ where $u=\int_x P$;
$$\therefore z=\int_x e^u\{\int_x e^{-u}Q+\phi y\}+f(y).$$

Ex. 1. Let $xy\dfrac{d^2z}{dx^2}=(n-1).y.\dfrac{dz}{dx}+a$;
$$\therefore z=\frac{x^n.\{\phi(y)\}^{n-1}}{n.(n-1)}-\frac{ax}{(n-1)y}+f(y).$$

Ex. 2. $xy\dfrac{d^2z}{dxdy}=bx\dfrac{dz}{dx}+ay$;
$$\therefore z=-\frac{ay\log x}{b-1}+\phi(x)+f(y).$$

137. To integrate $Rr+Ss+Tt=V$, where R, S, T and V are functions of x, y, z, p and q:

$$\because\ p=\frac{dz}{dx};\quad \therefore\ dp=\frac{d^2z}{dx^2}dx+\frac{d^2z}{dydx}dy=rdx+sdy,$$

$$\text{and } q=\frac{dz}{dy};\quad \therefore\ dq=\frac{d^2z}{dy^2}dy+\frac{d^2z}{dxdy}dx=tdy+sdx;$$

$$\therefore\ r=\frac{dp-sdy}{dx};\quad t=\frac{dq-sdx}{dy};\quad \therefore \text{ substituting}$$

$$Rdpdy+Tdqdx-Vdxdy=s(Rdy^2-Sdxdy+Tdx^2);$$

it is unnecessary to integrate the two members of this equation separately; for if we can integrate one of them so as to have the integral $N=a$, and by combining this with the other arrive at the integral $M=b$, M and N being functions of x, y, z, p and q, we may prove, as in a preceding article, that $M=\phi(N)$: and this result will give an equation with which we must proceed, as with an equation of partial differences of the first order.

Ex. 1. To integrate $\dfrac{d^2z}{dx^2}=c^2.\dfrac{d^2z}{dy^2}$; or $r=c^2t$:

$$\text{since } dp=rdx+sdy;\quad dq=tdy+sdx;$$

$$\therefore\ dpdy-c^2dqdx=s(dy^2-c^2dx^2);$$

$$\therefore\ \frac{dy^2}{dx^2}=c^2;\quad \therefore\ \frac{dy}{dx}=\pm c;\quad \therefore\ y-cx=a:\quad y+cx=a_1:$$

$$\text{and } dp.\frac{dy}{dx}-c^2dq=0;\quad \therefore\ cdp-c^2dq=0;$$

$$\therefore\ p-cq=b=\phi'(a)=\phi'(y-cx).$$

$$\text{But } p=\frac{dz-qdy}{dx};\quad \therefore\ dz-\phi'(y-cx)dx=q(dy+cdx);$$

$$\therefore\ dy+cdx=0;\quad \therefore\ y+cx=a,\ y=a-cx;$$

$$\therefore\ dz-\phi'(y-cx)\,dx=dz-\phi'(a-2cx)\,dx=0;$$

$$\therefore\ z-\phi(a-2cx)=\beta=f(a)=f(y+cx);$$

$$\therefore\ z=\phi(y-cx)+f(y+cx).$$

This is the equation of vibrating chords.

Ex. 2. Integrate $x^2r+y^2t+2xys=0$;

$$\therefore\ x^2dpdy+y^2dqdx=s.(x^2dy^2+y^2dx^2-2xydxdy);$$

$$\therefore\ (xdy - ydx)^2 = 0;\quad \therefore\ xdy - ydx = 0;\quad \therefore\ \frac{y}{x} = a = \frac{dy}{dx}$$

$$\text{and } \frac{x^2}{y^2} \cdot dp \cdot \frac{dy}{dx} + dq = 0;\quad dp + adq = 0;$$

$$\therefore\ p + aq = b = \phi(a) = \phi\left(\frac{y}{x}\right);\ \text{and } \because\ p = \frac{dz - qdy}{dx};$$

$$dz - dx\phi\left(\frac{y}{x}\right) = q(dy - adx) = 0;$$

$$\therefore\ dz - dx\phi(a) = 0;\ \therefore\ z - x\phi(a) = f(a);$$

$$\therefore\ z = x\phi\left(\frac{y}{x}\right) + f\left(\frac{y}{x}\right).$$

Ex. 3. Let $q^2r + p^2t - 2pqs = 0$; $y = \phi(z) - xf(z)$.

Ex. 4. Let $r + As + Bt + C = 0$;

$$z = -\frac{Cx^2}{2} + \phi(y - mx) + f(y - m_1x).$$

Ex. 5. $r - a^2t = xy$;

$$z = \frac{1}{6}x^3y + f(y + ax) + \phi(y - ax).$$

Ex. 6. $x^2r - y^2t = 0$;

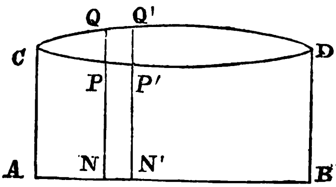
$$z = \sqrt{xy} \cdot f\left(\frac{y}{x}\right) + \phi xy.$$

CHAPTER VIII.

The Calculus of Variations.

1. IN the problems of Maxima and Minima hitherto solved, the form of the function which possessed the required property has always been given. But there is a class of problems, in which it is not only required to find when there is a maximum or minimum, but also the nature of the function which possesses the property.

Thus if the shortest distance between two given points be required, we must, in order to find the minimum distance, ascertain the nature of the curve that possesses the property; and thus, to continue the illustration, if C and D be the two points, we must select from the curves CPD, CQD, that which is the shortest: in our reasoning we must therefore pass from one curve to another curve, from a point P in one curve to a point Q in another curve: the change from P to Q is called a *variation* of the ordinate PN: the symbol of variation being δ.



Thus if $NP = y$, and if PQ be indefinitely small, the symbols δ of variation, and d of differentiation differ in this respect; by means of dy we pass from a point P to another P_1 indefinitely near to it, but in the same curve: by δy we transfer P to a point Q in another curve. We have here supposed the variation to be confined to the ordinate y: but $x = AN$ may also vary at the same time that y does.

2. Since $NP = y$; $\therefore N_1P_1 = y + dy$: also $NQ = y + \delta y$;

$$\therefore N_1Q_1 = NQ + d(NQ) = y + \delta y + d(y + \delta y),$$

$$\text{and } N_1Q_1 = N_1P_1 + \delta(N_1P_1) = y + dy + \delta(y + dy);$$

$$\therefore \delta y + d(y + \delta y) = dy + \delta(y + dy); \therefore d\delta y = \delta dy;$$

or the symbols of variation and differentiation are interchanged.

COR. Hence also if dy be put for y,

$$\therefore\ d\delta dy = \delta d^2 y;\quad \text{or} \because\ d\delta dy = dd\delta y = d^2\delta y;$$

$$\therefore\ d^2\delta y = \delta d^2 y, \text{ and thus } d^n\delta y = \delta d^n y.$$

If the differential of y be taken $= y_1 - y$, and the variation of y be also supposed to be very small, the theorem will admit of the following proof:

$$\delta dy = \delta(y_1 - y) = \delta y_1 - \delta y = d\delta y.$$

3. There is a similar theorem with regard to integration:

For let $\int u = u_1$; $\therefore\ u = du_1$; $\therefore\ \delta u = d\delta u_1$;

$$\therefore\ \int\delta u = \int\delta du_1 = \int d\delta u_1 = \delta u_1 = \delta\int u.$$

Also $\int^2\delta u = \int\delta\int u = \delta\int\int u = \delta\int^2 u$,

and thus $\int^n\delta u = \delta\int^n u$.

4. From the preceding we may perceive, that variation is only differentiation under a new symbol; and that to find the variation of a function of y, we must put $y + \delta y$ for y, and that term of the expanded function which involves δy will be the variation of the function required: or, what amounts to the same thing, the variation is the differential coefficient of $u = f(y)$ multiplied by δy; thus if

$u = y^n$, $\delta u = ny^{n-1}\delta y$; and if

$u = f(x, y, p, q, \text{\&c.})$ p, q, &c. being $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}$, &c., and

$$\therefore\ du = \frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dp}dp + \frac{du}{dq}dq + \text{\&c.}$$

$$= Mdx + Ndy + Pdp + Qdq + \text{\&c.}$$

by putting $M = \dfrac{du}{dx}$, $N = \dfrac{du}{dy}$, $P = \dfrac{du}{dp}$, &c.;

$$\therefore\ \delta u = M\delta x + N\delta y + P\delta p + Q\delta q + \text{\&c.}$$

5. To find $\delta\int u$, or $\int\delta u$; u being a function of y and x and their differentials; and y and x being dependent upon some other variable as s or t.

$$\therefore\ du = Mdx + Nd^2x + Pd^3x + Qd^4x + \text{\&c.}$$
$$+ mdy + nd^2y + pd^3y + qd^4y + \text{\&c.};$$

therefore $\because\ d^2x = ddx$, $d^3x = dd^2x$;

$$\therefore\ \delta u = M\delta x + N\delta dx + P\delta d^2x + Q\delta d^3x + \text{\&c.}$$
$$+ m\delta y + n\delta dy + p\delta d^2y + q\delta d^3y + \text{\&c.}$$

$$\therefore \int\delta u = \int(M\delta x + N\delta dx + P\delta d^2x + Q\delta d^3x + \&c.)$$
$$+ \int(m\delta y + n\delta dy + p\delta d^2y + q\delta d^3y + \&c.);$$

$\therefore$ integrating by parts, and placing δ after d,

$$\int N\delta dx = \int N d\delta x = N\delta x - \int dN\delta x,$$
$$\int P\delta d^2x = \int P d^2\delta x = P d\delta x - \int dP d\delta x$$
$$= P d\delta x - dP\delta x + \int d^2P\delta x,$$
$$\int Q\delta d^3x = \int Q d^3\delta x = Q d^2\delta x - \int dQ d^2\delta x$$
$$= Q d^2\delta x - dQ d\delta x + \int d^2Q d\delta x$$
$$= Q d^2\delta x - dQ d\delta x + d^2Q\delta x - \int d^3Q\delta x.$$
&c. &c. &c.

Similarly $\int n\delta dy = n\delta y - \int dn\delta y,$

$$\int p\delta d^2y = pd\delta y - dp\delta x + \int d^2p\delta x,$$
$$\int q\delta d^3y = qd^2\delta y - dqd\delta y + d^2q\delta y - \int d^3q\delta y,$$

and substituting these values in $\int\delta u$, we have

$$\int du = (N - dP + d^2Q - \&c.)\,\delta x + (n - dp + d^2q - \&c.)\,\delta y$$
$$+ (P - dQ + \&c.)\,d\delta x + (p - dq + \&c.)\,d\delta y$$
$$+ (Q - dR + \&c.)\,d^2\delta x + (q - dr + \&c.)d^2\delta y + \&c.$$
$$+ \int(M - dN + d^2P - d^3Q + \&c.)\,\delta x$$
$$+ \int(m - dn + d^2p - d^3r + \&c.)\,\delta y.$$

6. The result just obtained is composed of two similar parts, one due to the variation of δx, the other due to that of δy; and we thus see that had there been a third variable z, there must be added to the preceding expression, a series of terms similar to that which involves δx.

7. When u is of the form Vdx, to find the variation of $\int Vdx$.

Let $dV = Mdx + Ndy + Pdp + Qdq + Rdr + \&c.$

$$\text{where } p = \frac{dy}{dx};\ q = \frac{d^2y}{dx^2};\ r = \frac{d^3y}{dx^3};\ \&c.$$

$$\text{and } M = \frac{dV}{dx};\ N = \frac{dV}{dy};\ P = \frac{dV}{dp};\ \&c.$$

$$\therefore \delta V = M\delta x + N\delta y + P\delta p + Q\delta q + \&c.$$

Now $\delta\int Vdx = \int\delta\,.\,Vdx = \int(V\delta dx + dx\,.\,\delta V)$

$$= \int(Vd\delta x + dx\delta V) = \int Vd\delta x + \int dx\delta V$$
$$= V\delta x + \int(dx\delta V - \delta x dV).$$

But $\int(dx\delta V-\delta x dV)=\int dx\,(M\delta x+N\delta y+P\delta p+\&c.)$
$$-\int\delta x\,(Mdx+Ndy+Pdp+\&c.)$$
$$=\int_x N(\delta y-p\delta x)+\int_x P(\delta p-q\delta x)+\int_x Q(\delta q-r\delta x)+\&c.$$

But $\because\ p=\dfrac{dy}{dx};\quad q=\dfrac{dp}{dx};\quad r=\dfrac{dq}{dx};$

$$\therefore\ \delta p=\frac{dx\delta dy-dy\delta dx}{dx^2}=\frac{\delta dy-p\delta x}{dx};$$

$$\therefore\ \delta p-q\delta x=\frac{\delta dy-pd\delta x-dp\delta x}{dx}=\frac{d}{dx}(\delta y-p\delta x),$$

$$\delta q=\frac{dx\delta dp-dp\delta dx}{dx^2}=\frac{d\delta p-qd\delta x}{dx};$$

$$\therefore\ \delta q-r\delta x=\frac{d\delta p-qd\delta x-dq\delta x}{dx}=\frac{d}{dx}(\delta p-q\delta x).$$

Now let $\delta y-pdx=w;\quad\therefore\ \delta p-q\delta x=\dfrac{dw}{dx},$

$$\delta q-rdx=\frac{d^2w}{dx^2};\quad \delta r-sdx=\frac{d^3w}{dx^3};$$

$$\therefore\ \delta\int_x V=V\delta x+\int_x Nw+\int_x P\,.\frac{dw}{dx}+\int_x Q\,.\frac{d^2w}{dx^2}+\&c.$$

But $\int_x P\dfrac{dw}{dx}=Pw-\displaystyle\int_x\frac{dp}{dx}w,$

$$\int_x Q\frac{d^2w}{dx^2}=Q\frac{dw}{dx}-\int_x\frac{dQ}{dx}\,.\frac{dw}{dx}$$
$$=Q\frac{dw}{dx}-w\frac{dQ}{dx}+\int w\,.\frac{d^2Q}{dx^2};$$

and $\int_x R\,.\dfrac{d^3w}{dx^3}=R\dfrac{d^2w}{dx^2}-\dfrac{dR}{dx}\,.\dfrac{dw}{dx}+\dfrac{d^2R}{dx^2}w-\int w\,.\dfrac{d^3R}{dx^3};$

$$\therefore\ \delta\int_x V=V\delta x+\left(P-\frac{dQ}{dx}+\frac{d^2R}{dx^2}-\&c.\right)w$$
$$+\left(Q-\frac{dR}{dx}+\frac{d^2S}{dx^2}-\&c.\right)\frac{dw}{dx}$$
$$+\left(R-\frac{dS}{dx}+\&c.\right)\frac{d^2w}{dx^2}+\&c.$$
$$+\int_x\left(N-\frac{dP}{dx}+\frac{d^2Q}{dx^2}-\frac{d^3R}{dx^3}+\&c.\right)w.$$

8. Thus the variation of $\int_x V$ consists of two distinct parts, one of which is under the sign of integration, and the

other is not; the latter is affected only by the variations of the extreme values of y and x, the former is dependent upon all the values between those extreme ones. Let x_1, y_1, x_2, y_2 be the values of x and y at these limits, then the total variation of $\int_x V$

$$= V_2\delta x_2 - V_1\delta x_1 + (P_2 - \frac{dQ_2}{dx_2} + \frac{d^2R_2}{dx_2^2} - \&c.)\, w_2$$

$$- (P_1 - \frac{dQ_1}{dx_1} + \frac{d^2R_1}{dx_1^2} - \&c.)\, w_1 + \&c.$$

$$+ \int_x (N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \&c.)\, w.$$

The part under the sign of integration must be taken between the same limits.

9. If $u = f(xyz)$ and x still be the independent variable, then, since the differential of V, may be put

$$dV = Mdx + Ndy + Pdp + Qdq + \&c.$$
$$+ N'dz + P'dp' + Q'dq' + \&c.;$$
$$\therefore\ \delta V = M\delta x + Ndy + Pdp + Q\delta q + \&c.$$
$$+ N'dz + P'dp' + Q'\delta q' + \&c.$$

We shall have for the new variable z, a series of terms, involving N', P', Q', &c. similar to that in which N, P, Q, &c. are introduced; $\therefore$ if $\delta z - p'\delta x = w'$, the total variation will be expressed by

$$\delta \int_x V = V\delta x + (P - \frac{dQ}{dx} + \frac{d^2R}{dx^2} - \&c.)\, w$$

$$+ (P' - \frac{dQ'}{dx} + \frac{d^2R'}{dx^2} - \&c.)\, w'$$

$$+ (Q - \frac{dR}{dx} + \&c. \ldots)\frac{dw}{dx}$$

$$+ (Q' - \frac{dR'}{dx} + \&c. \ldots)\frac{dw'}{dx} + \&c.$$

$$+ \int_x (N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \&c.)\, w$$

$$+ \int_x (N_1 - \frac{dP_1}{dx} + \frac{d^2Q_1}{dx^2} - \&c.)\, w_1.$$

Maxima and Minima of Integral Formulæ.

10. We proceed to apply the results of the preceding article to the solution of some geometrical problems, involving the lengths and areas of curves, the surfaces and volumes of solids; when these quantities are, within certain limits of the variables, the greatest or least possible.

Now we know that if u, any function of x and y, be a maximum or minimum, $du = 0$; and by the same kind of reasoning which has been used to establish this proposition, it may be shewn that under the same circumstances the variation of u also vanishes; but if $u = Vdx$, we have seen that between the limits of x_1, y_1, x_2, y_2,

$$\delta \, . \int_x V = V_2 \delta x_2 - V_1 \delta x_1 + \left(P_2 - \frac{dQ_2}{dx_2} + \&c.\right) w_2$$

$$- \left(P_1 - \frac{dQ_1}{dx_1} + \&c.\right) w_1 + \&c.$$

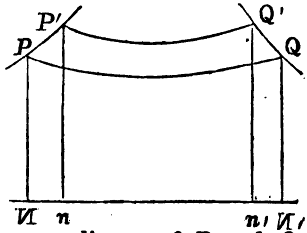
$$+ \int_x \left(N - \frac{dP}{dx} + \frac{d^2 Q}{dx^2} - \frac{d^3 R}{dx^3} + \&c.\right) w.$$

And since when $\int_x V$ is a maximum or minimum, $\delta \int_x V = 0$; therefore the two parts of which the variation of $\int_x V$ is composed must separately be put $= 0$; one part will determine the relations between the co-ordinates of the extreme values of the required function, the other the function which possesses the required maximum or minimum property.

11. Thus from $N - \dfrac{dP}{dx} + \dfrac{d^2 Q}{dx^2} - \&c. = 0$, and the equation $dV = Mdx + Ndy + Pdp + \&c.$ may be found the curve or function, which is the object of our enquiry, and from $V_2 \delta x_2 - V_1 \delta x_1 + P_2 w_2 - P_1 w_1$ &c. $= 0$, the position of its extreme points may be determined; if however the extreme points be fixed, $\delta x_1 = 0$, and $\delta x_2 = 0$, and the latter equation disappears.

12. Thus, if the shortest distance between two given points be required, the former equation will be quite sufficient for the problem; the constants of the integral being determined by the co-ordinates of the given points; but if we wish to find the shortest distance between two given curves the latter equation is also necessary, since it determines the points in the two curves to which the shortest distance is to be drawn.

Its use may be thus illustrated; let PP_1, QQ_1 be the given curves, and PQ, P_1Q_1 two curves drawn between them, and let P_1Q_1 be derived from PQ by writing $x+\delta x$, $y+\delta y$ for x and y: also let



$$\frac{dy}{dx}=m\,; \text{ and } \frac{dy}{dx}=n$$

be the equations to PP_1, and QQ_1; then if x_1y_1, x_2y_2 be the co-ordinates of P and Q; $\therefore \frac{\delta y_1}{\delta x_1}=m$; and $\frac{\delta y_2}{\delta x_2}=n$; (since the point P is always in PP_1 and Q in QQ_1;) and between these equations and that of the limits, $V_2\delta x_2-V_1\delta x_1+\&c.=0$; two of the quantities, as δy_1 and δy_2 may be eliminated, and the two independent variations, δx_1 and δx_2 will be left, the coefficients of which, being separately put $=0$, will give equations, by which and from the given equations to the curves, the points P and Q, may be determined.

We have here tacitly assumed that $\delta dy_1=0$ and $\delta dx_1=0$: if this be not the case, some new conditions must be fulfilled by the limits, which will enable us to introduce the higher differentials of the equations of the given curves: by means of which some of the variations δdx, δdy, &c. may be eliminated, and the coefficient of the remaining variations separately put $=0$, and the co-ordinates of the extreme points and the given conditions fulfilled.

13. We shall now deduce from the equation

$$N-\frac{dP}{dx}+\frac{d^2Q}{dx^2}-\&c.=0,$$

some formulæ of great use in the solution of Problems of maxima and minima.

Let $dV=Mdx+Ndy+Pdp+Qdq+Rdr+\&c.$,

$$\text{and } N-\frac{dP}{dx}+\frac{d^2Q}{dx^2}-\frac{d^3R}{dx^3}+\&c.=0.$$

(1) Let all but N and $P=0$;

$$\therefore \frac{dV}{dx}=Np+\frac{Pdp}{dx};\ \text{ and } N-\frac{dp}{dx}=0;\ \therefore N=\frac{dP}{dx};$$

$$\therefore \frac{dV}{dx}=p\frac{dP}{dx}+P\frac{dp}{dx};\ \therefore V=Pp+c.$$

(2) Let all but M, N, and P be $=0$;

$$\therefore \frac{dV}{dx}=M+\frac{d}{dx}(Pp+c); \quad \therefore V=\int_x M+Pp+c.$$

(3) Let $M=0$, $N=0$; and all the terms after $Q=0$;

$$\therefore \frac{dV}{dx}=P\frac{dp}{dx}+Q\frac{dq}{dx}; \quad \frac{dP}{dx}-\frac{d^2Q}{dx^2}=0; \quad \therefore P=\frac{dQ}{dx}+c;$$

$$\therefore \frac{dV}{dx}=q\frac{dQ}{dx}+Q\frac{dq}{dx}+c\frac{dp}{dx}; \quad \therefore V=Qq+cp+c_1.$$

Cor. If M does not $=0$; $V=Qq+cp+c_1+\int_x N$.

Prob. 1. Find the shortest distance between two given points in the same plane.

$$\text{Here } \int_x V=\int_x \sqrt{1+\frac{dy^2}{dx^2}}=\int_x \sqrt{1+p^2};$$

$$\therefore V=\sqrt{1+p^2}; \quad \therefore dV=\frac{p}{\sqrt{1+p^2}}dp;$$

$$\therefore M=0; \quad N=0; \quad P=\frac{p}{\sqrt{1+p^2}}, \quad Q=0.$$

$$\text{But } N-\frac{dP}{dx}+\&c.=0; \quad \therefore \frac{dP}{dx}=0; \quad \therefore P=c=\frac{p}{1+p^2};$$

$$\therefore p=\frac{c}{\sqrt{1-c^2}}=a; \quad \therefore y=ax+b,$$

the equation to a straight line; the constants a, b may be determined by the co-ordinates of the given points.

Prob. 2. Required the curve of quickest descent between two given points.

Let y be vertical and be measured downwards.

$$\text{Then time }=\int_x \frac{\frac{ds}{dx}}{\sqrt{2gy}}=\frac{1}{\sqrt{2g}}\int_x \frac{\sqrt{1+p^2}}{\sqrt{y}}=\frac{1}{\sqrt{2g}}\int_x V;$$

$$\therefore V=\frac{\sqrt{1+p^2}}{\sqrt{y}}, \quad dV=-\frac{\sqrt{1+p^2}}{2y^{\frac{3}{2}}}dy+\frac{p}{\sqrt{y}\sqrt{1+p^2}}dp;$$

$$\therefore M=0; \quad N=-\frac{\sqrt{1+p^2}}{2y^{\frac{3}{2}}}; \quad P=\frac{p}{\sqrt{y}\sqrt{1+p^2}}; \quad Q=0;$$

$$\therefore V=Pp+c; \quad \frac{\sqrt{1+p^2}}{\sqrt{y}}=\frac{p^2}{\sqrt{y}\sqrt{1+p^2}}+c;$$

$$\therefore \frac{1}{\sqrt{y}\sqrt{1+p^2}} = c = \frac{1}{\sqrt{2a}}; \quad \therefore \sqrt{1+p^2} = \sqrt{\frac{2a}{y}};$$

$$\therefore p = \sqrt{\frac{2a-y}{y}} \text{ the equation to the cycloid.}$$

PROB. 3. To find the shortest distance between two given curves.

From Prob. (1) $V = \sqrt{1+p^2}$; $p = c$; $y = ax + b$ the equation to the line which is the least distance required.

Let $\frac{dy}{dx} = m$, and $\frac{dy}{dx} = n$, be the equations to the two curves, and $y_1\ x_1$, $y_2\ x_2$ the co-ordinates of the points in which the shortest line intersects them; then since δy_1, δx_1, are the variations of y_1 and x_1 as we pass from one point to another adjacent one in the curve $\frac{dy}{dx} = m$;

$$\frac{\delta y_1}{\delta x_2} = \frac{dy_1}{dx_2} = m; \text{ and } \frac{\delta y_2}{\delta x_2} = \frac{dy_2}{dx_2} = n.$$

But $V_2\delta x_2 - V_1\delta x_1 + P_2 w_2 - P_1 w_1 = 0$,
whence since the variations of the extreme points $= 0$,

$$V_1\delta x_1 + P_1 w_1 = 0; \quad V_2\delta x_2 + P_2 w_2 = 0;$$

$$\therefore V_1\delta x_1 + P_1(\delta y_1 - p_1\delta x_1) = 0 \ldots (1);$$

$$V_2\delta x_2 + P_2(\delta y_2 - p_2\delta x_2) = 0 \ldots (2);$$

from (1) $V_1 + P_1 m - P_1 p_1 = 0$; $\therefore m = p_1 - \frac{V_1}{P_1} = -\frac{1}{p_1} = -\frac{1}{c}$;

from (2) $V_2 + P_2 n - P_2 p_2 = 0$; $\therefore n = p_2 - \frac{V_2}{P_2} = -\frac{1}{p_2} = -\frac{1}{c}$;

$$\therefore 1 + cm = 0; \text{ and } 1 + cn = 0;$$

which shew that the line must cut both curves at right angles. Also the equation to the line being

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_2}(x - x_1);$$

$\therefore c = \frac{y_2 - y_1}{x_2 - x_1}$; whence substituting for c, in $1 + cm = 0$;

and $1 + cn = 0$; we shall have with the equations to the given curves four equations to determine the four quantities, y_1, x_1, y_2, x_2, and thus the line is completely determined.

PROB. 4. Find the curve, which within its own arc, its evolute and radius of curvature, contains the least area.

$$\text{Here } \frac{dA}{dx} = \frac{R}{2}\frac{ds}{dx} = \frac{(1+p^2)^{\frac{3}{2}}}{-2q}\sqrt{1+p^2} = \frac{(1+p^2)^2}{-2q};$$

$$\therefore V = \frac{(1+p^2)^2}{q}; \text{ whence the curve is the cycloid.}$$

PROB. 5. Find the curve when $\int_x \frac{p^3 y}{1+p^2}$ is a minimum. This is the solid of least resistance,

$$y = \frac{c(1+p^2)^2}{p^3}; \quad x = c_1 + c\left(\frac{3}{4p^4} + \frac{1+p^2}{p^2} + \log p\right).$$

PROB. 6. Find the curve of quickest descent from one curve to another curve, the velocity being that from a horizontal line.

Here $V = \frac{\sqrt{1+p^2}}{\sqrt{y}}$; $\therefore p = \frac{\sqrt{2a-y}}{y}$, the equation to the cycloid; the equations of the limits give

$$V_1\delta x_1 + P_1\delta y_1 - P_1 p_1 \delta x_1 = 0; \quad V_2\delta x_2 + P_2\delta y_2 - P_2 p_2 \delta x_2 = 0,$$

from which since $\frac{\delta y_1}{\delta x_1} = m$, and $\frac{\delta y_2}{\delta x_2} = n$, we have

$$V_1 - P_1 p_1 + P_1 m = 0; \text{ and } V_2 - P_2 p_2 + P_2 n = 0.$$

$$\text{But } V_1 - P_1 p_1 = c = \frac{1}{\sqrt{2a}}, \quad P = \frac{p}{\sqrt{y}\sqrt{1+p^2}} = \frac{p}{\sqrt{2a}};$$

$$\therefore \frac{1}{\sqrt{2a}} + \frac{p_1 m}{\sqrt{2a}} = 0; \text{ and } \frac{1}{\sqrt{2a}} + \frac{p_2 n}{\sqrt{2a}} = 0;$$

$$\therefore 1 + p_1 m = 0; \text{ and } 1 + p_2 n = 0,$$

which equations shew that the cycloid cuts both the curves at right angles, and from $p = \sqrt{\frac{2a-y}{y}}$, we see that the base of the cycloid coincides with the horizontal line from which we have supposed the body to have commenced its motion.

PROB. 7. To find the curve of quickest descent from one given curve to another given curve, the motion commencing from the upper curve.

Let y_1 be the value of the ordinate at the point at which the motion commences, y the value at the end of time t.

Then time $= \frac{1}{\sqrt{2g}} \int_x \frac{\sqrt{1+p^2}}{\sqrt{y-y_1}}$; $\therefore V = \frac{\sqrt{1+p^2}}{\sqrt{y-y_1}}$.

In this problem the function V involves y_1 one of the variable co-ordinates of the limits; in such a case we must add the term $\delta y_1 \int_x \frac{dV}{dy}$ to the equation of the limits, and then the whole variation of the $\int_x V$ will become

$$\delta \int_x V = V_2 \delta x_2 - V_1 \delta x_1 + \delta y_1 \int_x \frac{dV}{dx_1} + P_2 w_2 - P_1 w_1 + \&c.$$

$$+ \int_x \left(N - \frac{dP}{dx} + \frac{d^2 Q}{dx^2} - \&c.\right) w.$$

Now referring to the problem,

$$dV = -\frac{\sqrt{1+p^2}}{2(y-y_1)^{\frac{3}{2}}} dy + \frac{pdp}{\sqrt{y-y_1}\sqrt{1+p^2}} + \frac{\sqrt{1+p^2}}{2(y-y_1)^{\frac{3}{2}}} dy_1;$$

$$\therefore N = -\frac{\sqrt{1+p^2}}{2(y-y_1)^{\frac{3}{2}}};\ P = \frac{p}{\sqrt{y-y_1}\sqrt{1+p^2}};\ \frac{dV}{dy^1} = \frac{\sqrt{1+p^2}}{2(y-y_1)^{\frac{3}{2}}},$$

whence from the formula $V = Pp + c$,

$$\frac{\sqrt{1+p^2}}{\sqrt{y-y_1}} = \frac{p^2}{\sqrt{y-y_1}\sqrt{1+p^2}};\quad \therefore \sqrt{y-y_1}\sqrt{1+p^2} = \frac{1}{C} = \sqrt{2a},$$

the equation to a cycloid, the cusp being at the point from which the motion commences.

Next, to find $\int_x \frac{dV}{dy_1}$; $\frac{dV}{dy_1} = -N = -\frac{dP}{dx}$, $\because N - \frac{dP}{dx} = 0$;

$$\therefore \int_x \frac{dV}{dy_1} = -P;\quad \therefore \int_{x_1}^{x_2} \frac{dV}{dx} = P_1 - P_2;$$

$$\therefore V_2 \delta x_2 - V_1 \delta x_1 + P_2 w_2 - P_1 w_1 + (P_1 - P_2)\delta y_1 = 0,$$

$$\text{or } (V_2 - P_2 p_2)\delta x_2 - (V_1 - P_1 w_1)\delta x_1 + P_2 \delta y_2 - P_2 \delta y_1 = 0;$$

$$\therefore \frac{1}{\sqrt{2a}}(\delta x_2 - \delta x_1) + p_2 \delta y_2 - p_2 dy_1 = 0.$$

But if $\frac{dy}{dx} = m$ and $\frac{dy}{dx} = n$, be the equations to the given curves, $\frac{\delta y_1}{\delta x_1} = m$ and $\frac{\delta y_2}{\delta x_2} = n$; $\therefore$ substituting

$$(1 + p_2 n)\delta x_2 - (1 + p_2 m)\delta x_1 = 0,$$

whence $\because$ δx_2 and δx_1 are independent variables,

$$1 + p_2 n = 0;\quad 1 + p_2 m = 0;\quad \therefore m = n,$$

or the tangents at the points where the cycloid cuts the two curves are parallel; also since $p_2 = -\frac{1}{n}$, it cuts the second curve at right angles.

14. We shall now consider problems in which z is also a function of x, and shall make use of the general method investigated in Art. 5.

PROB. 1. Required the shortest distance between two points in space.

Let s be the distance; $\therefore \int ds = \int \sqrt{dx^2 + dy^2 + dz^2}$;

$$\therefore \delta \int ds = \int \delta ds = \int \frac{dx}{ds} \delta dx + \int \frac{dy}{ds} \delta dy + \int \frac{dz}{ds} \delta dz$$

$$= \frac{dx}{ds} \delta x + \frac{dy}{ds} \delta y + \frac{dz}{ds} \delta z - \int \{ d \cdot \frac{dx}{ds} \delta x + d \cdot \frac{dy}{ds} \cdot \delta y + d \cdot \frac{dz}{ds} \delta z \},$$

whence $M = 0$, $m = 0$, $M' = 0$;

$$N = \frac{dx}{ds}; \quad n = \frac{dy}{ds}; \quad N' = \frac{dz}{ds}; \quad dN = 0 = dn = dN';$$

$$\therefore \frac{dx}{ds} = a; \quad \frac{dy}{ds} = b; \quad \frac{dz}{ds} = c.$$

Also $\frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} + \frac{dz^2}{ds^2} = 1 = a^2 + b^2 + c^2$;

an equation which connects the constants a, b, c;

also $\because \frac{dx}{dz} = \frac{a}{c}$; $\frac{dy}{dz} = \frac{b}{c}$; $\therefore x = \frac{a}{c} z + c'$; $y = \frac{b}{c} z + c''$,

the equations to the projections of a straight line.

PROB. 2. Find the equation to the shortest line that can be drawn upon a given curve surface.

Let $dz = pdx + qdy$ be the equation of the surface; then the variations of the co-ordinates x, y, z which are under the sign $\int$ must satisfy the differential equation; $\therefore \delta z = p\delta x + q\delta y$;

$$\therefore \delta \int ds = \left(\frac{dx}{ds} + p \frac{dz}{ds} \right) \delta x + \left(\frac{dy}{ds} + q \cdot \frac{dz}{ds} \right) \delta y$$

$$- \int \left\{ \left(d \cdot \frac{dx}{ds} + p \cdot d \cdot \frac{dz}{ds} \right) \delta x + \left(d \cdot \frac{dy}{ds} + q \cdot d \cdot \frac{dz}{ds} \right) \delta y \right\},$$

whence we have from the part under the sign of integration

$$d \cdot \frac{dx}{ds} + p \cdot d \cdot \frac{dz}{ds} = 0; \text{ and } d \cdot \frac{dy}{ds} + q \cdot d \cdot \frac{dz}{ds} = 0;$$

whence, having found p and q from the equation to the given surface, the equations to the curve may be found.

Ex. 1. Let the surface be a surface of revolution;

$$\therefore\ z = \phi(x^2 + y^2);$$

$$\therefore\ p = 2x\phi'.(x^2+y^2);\quad q = 2y.\phi'(x^2+y^2);$$

and considering s to be the independent variable,

$$\therefore\ \frac{d^2x}{ds^2} + p.\frac{d^2z}{ds^2} = \frac{d^2x}{ds^2} + 2x.\phi'(x^2+y^2)\frac{d^2z}{ds^2} = 0, \quad (1)$$

$$\text{and}\ \frac{d^2y}{ds^2} + q.\frac{d^2z}{ds^2} = \frac{d^2y}{ds^2} + 2y.\phi'(x^2+y^2)\frac{d^2z}{ds^2} = 0, \quad (2)$$

$$(1)\times y - (2)\times x;\quad \therefore\ y\frac{d^2x}{ds^2} - x\frac{d^2y}{ds^2} = 0.$$

But if $x^2 + y^2 = r^2$, and $\theta = \cos^{-1}\dfrac{x}{r}$;

$$yd^2x - xd^2y = d.(r^2d\theta);$$

$$\therefore\ \text{integrating,}\ r^2\frac{d\theta}{ds} = c;\quad \therefore\ \frac{rd\theta}{ds} = \frac{c}{r}.$$

But $\dfrac{rd\theta}{ds}$ is the sine of the angle at which the shortest line cuts the generating curve or meridian, hence if ϕ be this angle,

$$\sin\phi = \frac{c}{r} = \frac{c}{\sqrt{x^2+y^2}}.$$

Cor. Since $r^2\dfrac{d\theta}{ds} = c$;

$$\therefore\ r^4\frac{d\theta^2}{dr^2} = c^2\frac{ds^2}{dr^2} = c^2\left(1 + r^2\frac{d\theta^2}{dr^2} + \frac{dz^2}{dr^2}\right);$$

$$\therefore\ r^2(r^2 - c^2)\frac{d\theta^2}{dr^2} = c^2\{1 + \phi'(r)^2\};$$

$$\therefore\ \frac{d\theta}{dr} = \frac{c}{r}\sqrt{\frac{1+\phi'(r)^2}{r^2-c^2}},$$

the equation to the projection on the plane of xy.

Ex. 2. Let the surface be a sphere.

Then if $a\psi$ be the distance of the point of intersection of the shortest line with the meridian, from the point where the axis of z meets the sphere, $r = a\sin\psi$;

$$\therefore\ \sin\phi = \frac{c}{a\sin\psi};\quad \sin\phi.\sin\psi = \frac{c}{a},$$

or is constant, which is a property of an arc of a great circle. Hence the shortest line is the arc of a great circle.

Ex. 3. If the surface be a cone,

$$z = e\sqrt{x^2+y^2} = er; \quad \frac{dz}{dr} = e;$$

$$\therefore \frac{d\theta}{dr} = \frac{c}{r}\frac{\sqrt{1+e^2}}{\sqrt{r^2-c^2}};$$

$$\therefore \theta + c_1 = \sqrt{1+e^2}\sec^{-1}\left(\frac{r}{c}\right); \quad \therefore r = c\sec\left(\frac{\theta+c_1}{\sqrt{1+e^2}}\right),$$

the equation to the elliptic spiral.

15. Resuming the equations, Art. 14, Prob. 2, since

$$\frac{d^2x}{ds^2} = -p\cdot\frac{d^2z}{ds^2},$$

$$\frac{d^2y}{ds^2} = -q\cdot\frac{d^2z}{ds^2},$$

square, add, and add $\left(\frac{d^2z}{ds^2}\right)^2$;

$$\therefore \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2 = (1+p^2+q^2)\left(\frac{d^2z}{ds^2}\right)^2;$$

or if R be the radius of absolute curvature, and γ the angle which the normal to the curve makes with the axis of z,

$$\therefore \frac{1}{R^2} = \frac{1}{\cos^2\gamma}\cdot\left(\frac{d^2z}{ds^2}\right)^2; \quad \therefore \pm\frac{d^2z}{ds^2} = \frac{\cos\gamma}{R}.$$

But if $\frac{dz}{ds}$ be the direction cosine of the angle which the tangent makes with z,

$$\therefore \frac{dz}{ds} = \sin\gamma; \quad \frac{d^2z}{ds^2} = \cos\gamma\cdot\frac{d\gamma}{ds} = \pm\frac{\cos\gamma}{R};$$

$$\therefore \frac{1}{R} = \pm\frac{d\gamma}{ds}.$$

Now we know, if θ be the angle which the tangent to a plane curve makes with the axis of x, that $\pm\frac{d\theta}{ds} = \frac{1}{R}$, or that the rate at which the inclination of the tangent increases or decreases, varies as the curvature, hence the preceding result shews that this is also true for the curve of double curvature under consideration, possessing the property of being the minimum distance between any two points on a surface.

Isoperimetrical Problems.

16. The preceding problems have been those of absolute maxima and minima; we now treat of questions of relative maxima and minima. Of this kind is the problem, 'Given the length of a curve, find its equation, when the area included by it is a maximum.' This Problem was first proposed and solved by James Bernoulli, and from its nature was called Isoperimetrical, a name which was extended to all problems of this kind.

The problem of relative maxima and minima may be defined to be this. 'Find $y=f(x)$; so that $\int_x u$ may be a maximum while $\int_x u_1 = c$.' To solve this problem, we multiply $\int_x u_1$ by a constant a, and add the product to $\int_x u$, and make $\delta . \{\int_x u + a\int_x u_1\} = 0$; or $\delta \int_x (u + au_1) = 0$; for since $\int_x u$ is a maximum and $\int_x u_1 = c$, their separate variations will $=0$; and as the former is limited by the values of the variables in the latter, all the conditions of the problem will be fulfilled and included in the variation of $\int_x (u + au_1)$; so also if there be another equation of condition $\int_x u_2 = c_1$, we must add $b\int_x u_2$ to the former integrals, and take the variation of $\int_x (u + au_1 + bu_2)$.

Hence instead of V, we must write $V+au_1$ or $V+au_1+bu_2$ in $\delta\int_x V$, and then proceed as in absolute maxima and minima: in these examples the total variation will be expressed by $\delta\int_x V_1$.

PROB. 1. Of all curves of equal perimeters, find that which has the greatest area.

$$\text{Here } \int_x u_1 = \int_x \sqrt{1+p^2} = c; \quad \int_x V = \int_x y;$$

$$\therefore \int_x V_1 = \int_x (y + a\sqrt{1+p^2}); \quad \therefore V_1 = y + a\sqrt{1+p^2};$$

$$\therefore dV_1 = dy + \frac{ap}{\sqrt{1+p^2}}; \quad \therefore N = 1, \; P = \frac{ap}{\sqrt{1+p^2}}; \quad Q = 0;$$

$$\therefore V_1 = Pp + C; \quad \therefore y + a\sqrt{1+p^2} = \frac{ap^2}{\sqrt{1+p^2}} + c;$$

$$\therefore y - c = -\frac{a}{\sqrt{1+p^2}}; \quad \therefore p = \frac{\sqrt{a^2-(y-c)^2}}{y-c};$$

$$\therefore \frac{dx}{dy} = \frac{y-c}{\sqrt{a^2-(y-c)^2}}; \quad x - c_1 = -\sqrt{a^2-(y-c)^2};$$

$$\therefore (x-c_1)^2 + (y-c)^2 = a^2. \text{ The equation to the circle.}$$

PROB. 2. Find the curve in which a chain of given